

L^2 -estimates for a scattering transform in two
dimensions

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A system in two dimensions

In this talk, we consider the 2×2 -system in the plane:

$$\begin{pmatrix} \partial_{\bar{x}} & 0 \\ 0 & \partial_x \end{pmatrix} \psi - \begin{pmatrix} 0 & q^1 \\ q^2 & 0 \end{pmatrix} \psi = 0.$$

Though it offends my friends in complex analysis, we use $x = x^1 + ix^2$ to denote a complex variable and thus $\partial_{\bar{x}}$ and ∂_x are the standard derivatives with respect to \bar{x} and x .

The solution ψ will be a 2×1 vector or 2×2 matrix.

We write the system more compactly as

$$(D - Q)\psi = 0$$

where

$$D = \begin{pmatrix} \partial_{\bar{x}} & 0 \\ 0 & \partial_x \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & q^1 \\ q^2 & 0 \end{pmatrix}$$

Why is this interesting?

This system was originally studied by Beals, Coifman (1985,1988), Fokas, Ablowitz (1984) and Sung (1994) because it is connected to solving a non-linear evolution equation in (2+1) dimensions, the Davey-Stewartson II system,

$$\begin{cases} q_t = iq_{x^1x^2} - 4irq \\ r_{x^1x^1} + r_{x^2x^2} = (|q|^2)_{x^1x^2} \end{cases}$$

by the inverse scattering method.

Why is this interesting? (continued)

If we consider a solution u the conductivity equation,

$$\operatorname{div} \gamma(x) \nabla u(x) = 0,$$

then

$$\begin{pmatrix} v \\ w \end{pmatrix} = \gamma^{1/2} \begin{pmatrix} \partial_x u \\ \partial_{\bar{x}} u \end{pmatrix}$$

satisfies

$$(D - Q) \begin{pmatrix} v \\ w \end{pmatrix} = 0.$$

where the potential Q is related to γ by

$$Q = \begin{pmatrix} 0 & -\partial_x \log \sqrt{\gamma} \\ -\partial_{\bar{x}} \log \sqrt{\gamma} & 0 \end{pmatrix}.$$

Relation to inverse conductivity problem.

With Uhlmann (1997), we used the system $D-Q$ to study the *inverse conductivity problem* and showed that if $\nabla\gamma$ is in $L^p(\Omega)$ for some $p > 2$, then γ is uniquely determined by the Dirichlet to Neumann map.

Recall that the Dirichlet to Neumann map, Λ_γ is the map given by

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}$$

where u is the solution of the Dirichlet problem

$$\begin{cases} \operatorname{div} \gamma \nabla u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Of course, this extended the work of Nachman (1996) who was the first to prove uniqueness in two dimensions, but required that the coefficient have two derivatives.

Scattering for the first order system.

Consider the family of solutions of the free system

$$D\psi_0 = 0$$

which are parameterized by the complex variable $z \in \mathbf{C}$:

$$\psi_0(x, z) = \begin{pmatrix} e^{ixz} & 0 \\ 0 & e^{-i\bar{x}z} \end{pmatrix}$$

We look for solutions of $(D - Q)\psi = 0$ which are of the form

$$\psi(x, z) = m(x, z)\psi_0(x, z)$$

with

$$m = 1 \quad \text{at infinity.}$$

Scattering for a first order system (continued)

A calculation shows that the 2×2 matrix, m , should satisfy

$$(D_z - Q)m = 0$$

where the operator D_z is given by

$$D_z f(x, z) = E_z^{-1} D E_z f.$$

The map E_z acts on the diagonal part of f , f^d and the off-diagonal part of f , f^o , by

$$E_z f(x, z) = f^d(x, z) + A(x, z)^{-1} f^o(x, z).$$

The matrix $A(x, z)$ is defined by

$$A(x, z) = \begin{pmatrix} a^1(x, z) & 0 \\ 0 & a^2(x, z) \end{pmatrix}$$

$$a^1(x, z) = \exp(ix\bar{z} + i\bar{x}z) \quad a^2(x, z) = a^1(x, -\bar{z}).$$

Since the exponents in a^1 and a^2 are purely imaginary, we have that

$$\|E_z f\|_p = \|f\|_p.$$

Some simple estimates

We let $G_z = D_z^{-1} = E_z^{-1}GE_z$, where G is defined by

$$Gf(x) = \int_{\mathbf{C}} \frac{f(y)}{x-y} d\mu(y).$$

Fractional integration. If $1 < p < 2$, then

$$f \rightarrow G_z f \quad \text{maps } L^p(\mathbf{C}) \rightarrow L^{p^*}(\mathbf{C})$$

where $1/p^* = 1/p - 1/2$.

Hölder's inequality. If $Q \in L^2$, then

$$f \rightarrow Qf \quad \text{maps } L^{p^*}(\mathbf{C}) \rightarrow L^p(\mathbf{C}).$$

Construction of m .

Thus, if $Q \in L^p(\mathbf{C})$ for some p between 1 and 2 and $Q \in L^2(\mathbf{C})$ with $\|Q\|_{L^2}$ sufficiently small, then we can construct m as the series

$$m(\cdot, z) = 1 + \sum_{j=1}^{\infty} (G_z Q)^j (G_z Q)(\cdot).$$

The sum will converge in L^{p^*} .

When $p = 4/3$ and $p^* = 4$, we can show convergence when

$$\int_{\mathbf{C}} |q^1(x)|^2 + |q^2(x)|^2 d\mu(x) < 2$$

thanks to E. Lieb's (1983) sharp estimates for fractional integration.

If $Q = Q^*$, we can construct m without requiring that Q be small. But that is another talk.

The $\frac{\partial}{\partial \bar{z}}$ equation

With enough assumptions on Q , we can differentiate the solution m with respect to the parameter z and obtain that

$$\frac{\partial}{\partial \bar{z}} m(x, z) = m(x, \bar{z}) S(z) A(x, -\bar{z})$$

where the scattering data S is given by

$$S(z) = -\frac{2J}{\pi} \int_{\mathbf{C}} E_z(Q(x)m(x, z))^o d\mu(x)$$

and the matrix J is defined by

$$J = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

In addition to appearing in the $\frac{\partial}{\partial \bar{z}}$ -equation, $S(z)$ is scattering data in the traditional sense that it appears in the asymptotic expansion of m as x approaches ∞ .

The scattering map

The map

$$Q \rightarrow S$$

is the scattering map.

The properties of this map (and its inverse) have consequences for the inverse conductivity problem and for non-linear evolution equations.

For example, in the inverse conductivity problem, we can determine S from the Dirichlet to Neumann map. If we know S is in L^2 , then we can use the $\partial/\partial\bar{z}$ equation to show that m is uniquely determined by S .

Some properties of the scattering map.

The map $Q \rightarrow S$ shares many properties with the Fourier transform.

The map $Q \rightarrow S$ takes potentials in the Schwartz class to scattering data in the Schwartz class (Beals, Coifman (1985), Sung(1994)).

Beals and Coifman (1988) have shown that if we have $Q = Q^*$ and Q is nice, then we have a version of the Plancherel theorem

$$\int |Q|^2 d\mu = \int |S|^2 d\mu.$$

The proof is magic and does not give continuity properties of the scattering map.

Barceló, Barceló and Ruiz (1999) have established continuity of the map $Q \rightarrow S$ when Q is Hölder continuous and compactly supported. This is one step in establishing the continuous dependence of γ on Λ_γ in the inverse conductivity problem.

The main result.

Theorem. (B. 2000) The map $Q \rightarrow S$ and the inverse map $S \rightarrow Q$ extend continuously to the set of 2×2 matrix-valued functions whose diagonal entries are zero and for which the L^2 norm is at most $\sqrt{2}$.

A sketch of the proof.

The proof is fairly straightforward. We substitute the series for m

$$m(\cdot, z) = 1 + \sum_{j=1}^{\infty} (G_z Q)^j (G_z Q)(\cdot)$$

into the definition of the scattering data,

$$S(z) = -\frac{2J}{\pi} \int_{\mathbf{C}} E_z(Q(x)m(x, z))^o d\mu(x)$$

and express S as the series

$$\begin{aligned} S(z) &= \frac{-2}{\pi} J \int_{\mathbf{C}} A(x, -z) Q(x) \\ &\quad \times \sum_{j=0}^{\infty} (GQ G_z Q)^j (1) d\mu(x) \\ &\equiv \frac{-2}{\pi} J \sum_{j=0}^{\infty} S_j(z). \end{aligned}$$

Notice that the first term ($j = 0$) is essentially the Fourier transform of S .

A sketch of the proof. (continued)

We consider one entry in the j th term of the series for S .

$$S_k^{12}(z) = \frac{1}{\pi^{2k}} \int_{\mathbf{C}^{2k+1}} a^1(-x_0 + x_1 - \dots - x_{2k}, z) \\ \times \frac{Q^{12}(x_0) \dots Q^{12}(x_{2k})}{(\bar{x}_0 - \bar{x}_1) \dots (x_{2k-1} - x_{2k})} d\mu(x_0, \dots, x_{2k}).$$

We estimate this by duality. Choose a sufficiently nice function T , then we have

$$\int_{\mathbf{C}} T(z) S_k^{12}(z) d\mu(z) = \frac{1}{\pi^{2k}} \int_{\mathbf{C}^{2k+1}} \\ \hat{T}(2(x_0 - x_1 + x_2 - \dots - x_{2k-1} + x_{2k})) \\ \times \frac{Q^{12}(x_0) Q^{21}(x_1) \dots Q^{21}(x_{2k-1}) Q^{12}(x_{2k})}{(\bar{x}_0 - \bar{x}_1)(x_1 - x_2) \dots (x_{2k-1} - x_{2k})} \\ d\mu(x_0, \dots, x_{2k}).$$

The main estimate.

Thus, we need to consider multi-linear expressions of the form:

$$\begin{aligned} & I_k(t, q_0, \dots, q_{2k}) \\ &= \int_{\mathbf{C}^{2k+1}} \frac{t(x_0 - x_1 + x_2 - \dots - x_{2k-1} + x_{2k})}{|x_0 - x_1| |x_1 - x_2| \cdots |x_{2k-1} - x_{2k}|} \\ & \quad \times \prod_{j=0}^{2k} q_j(x_j) d\mu(x_0, \dots, x_{2k}). \end{aligned}$$

The main technical estimate is

Lemma For every $\epsilon > 0$, there exists a constant C_ϵ so that

$$I_k(t, q_0, \dots, q_{2k}) \leq C_\epsilon \pi^{2k} (1 + \epsilon)^{2k} \|t\|_2 \prod_{j=0}^{2k} \|q_j\|_2.$$

The end of the proof.

This estimate allows us to sum the series for S in $L^2(\mathbf{C})$.

A bit more work, gives the continuous dependence.

Finally, the inverse map $S \rightarrow Q$ is of a similar form, and can be studied in the same way.

Some questions.

1. Can we construct the solutions m when the potential Q is in L^2 ? The above argument estimates the scattering map but does not directly construct the Jost solutions, m .
2. Can we establish uniqueness in the inverse conductivity problem when the coefficient γ has only one derivative in L^2 ?
3. In the plane, do we have continuous dependence of the conductivity on the Dirichlet to Neumann map when the conductivity has one derivative in L^p , $p > 2$? $p = 2$?

Some questions (continued).

4. Can we establish uniqueness for the inverse conductivity problem in higher dimensions when the conductivity has only one derivative in L^p for some $p > n$? The best result in higher dimensions shows that we have uniqueness when the conductivity has $3/2$ derivatives. This is due to Panchenko, Paivarinta and Uhlmann (2000). Also, a manuscript in preparation of the speaker and R. Torres, will show uniqueness when the conductivity has $3/2$ derivatives in L^p for $p > 2n$.

5. Can one construct examples of non-uniqueness? That is, can we find two conductivities which have the same Dirichlet to Neumann map?

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