Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in L^p for $p > 2n$.

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The inverse conductivity problem.

Let $\gamma : \Omega \to (0, \infty)$ be a scalar-valued function such that div$\gamma\nabla$ is a strictly elliptic operator. Consider the Dirichlet problem for this operator in a domain Ω with reasonable boundary:

$$\begin{cases} \text{div}\gamma\nabla u = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases}$$

The inverse conductivity problem is the problem of recovering the coefficient γ from knowledge of the map Λ_γ which takes Dirichlet to Neumann data:

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}.$$

My particular interest is to find what regularity conditions the coefficient must satisfy in order to prove uniqueness. This may lead to interesting questions in harmonic analysis.

A related equation.

If we set $v = \sqrt{\gamma} u$, then we have that v satisfies the Schrödinger equation,

$$\Delta v - qv = 0.$$

where the potential q is given by

$$q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}.$$

We will see how to construct a large family of solutions to this equation.

Some history.

Using this reduction to Schrödinger operators, a number of authors have proven that the map $\gamma \to \Lambda_\gamma$ is injective, provided,

Sylvester and Uhlmann, 1987, $n \geq 3$, γ is C^2

Novikov, 1988, $n \geq 3$, γ smooth.

Nachman, 1988, $n \geq 3$, γ C^2.

Ramm, 1989, $n = 3$, γ with two derivatives in L^2.

Chanillo, 1990, $n \geq 3$, the analagous result for Schrödinger operators with potentials in the Fefferman-Phong class, $F^{(n-1)/2}$.

Nachman, 1996, $n = 2$, provided γ has two derivatives in L^p, $p > 1$.

Brown, 1996, $n \geq 3$, γ is of class $C^{3/2+\epsilon}$.

Brown and Uhlmann, 1997, $n = 2$, γ has one derivative in L^p, $p > 2$.

Paivarinta, Panchenko, Uhlmann, 2000, $n \geq 3$, γ is of class $C^{3/2}$.

Lassas, Greenleaf, Uhlmann, 2002, $n \geq 3$, γ smooth off of a hypersurface and globally $C^{1+\epsilon}$.

The main technique.

It is comparatively easy to recover the boundary values of γ and derivatives of γ from the Dirichlet to Neumann map. Thus, if we have two coefficients γ_1 and γ_2, with the same Dirichlet to Neumann map, we can extend γ_j to $\mathbf{R}^n$, preserving smoothness and so that $\gamma_1 = \gamma_2$ outside of Ω.

We make the reduction to a Schrödinger operator, discussed above and then, based on an idea of Calderón, we try to construct solutions of the Schrödinger operator which approximate harmonic exponentials,

$$v(x) = e^{x\cdot\zeta}(1 + \psi(x)),$$

where $\zeta \in \mathbf{C}^n$ satisfies $\zeta \cdot \zeta = 0$.

Thus, the function ψ should satisfy

$$\Delta\psi + 2\zeta \cdot \nabla\psi - q\psi = q.$$

Construction of exponentially growing solutions. We construct solutions of the equation:

$$\Delta\psi + 2\zeta \cdot \nabla\psi - q\psi = q$$

in the form

$$\psi = \sum_{j=0}^{\infty} (G_\zeta m_q)^j(1)$$

where m_q is the multiplication operator given by the distribution q.

Thus, we need:

1. Estimates for the operator $G_\zeta = (\Delta + 2\zeta \cdot \nabla)^{-1}$.

2. Estimates for the bilinear map $q, \psi \to q\psi$.

The estimate for G_ζ, I.

The estimate we use is a simple consequence of the work of Sylvester and Uhlmann. We have a little bit new to say about the second point, where we essentially use the Sobolev embedding theorem to conclude that product is slightly better than one might first expect.

It is well-known (see Sylvester and Uhlmann) that if $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$, then

$$\int_{\mathbf{R}^n} (|\zeta|^2 |G_\zeta f|^2 + |\nabla G_\zeta f|^2)(1+|x|^2)^{-1}\, dx \qquad (1)$$
$$\leq \; C(n) \int_{\mathbf{R}^n} (|f|^2 (1+|x|^2)^1\, dx.$$

The estimates for G_ζ, II.

If we define spaces B_s^* by

$$\|f\|_{B_s^*} \leq \sup_{R>1} R^{-1/2} \|\eta_R f\|_{W^{s,2}}$$

where η is a function which is one on the unit ball and compactly supported in a the ball of radius 2, centered at the origin.

Then, the estimate (1), duality, and interpolation imply.

Theorem 1 *If $0 \leq s \leq 1$, f is compactly supported in a ball of radius R_0, then*

$$\|G_\zeta f\|_{B_{1/2}^*} \leq \frac{C R_0^{1/2}}{|\zeta|^s} \|f\|_{W^{-1/2+s,2}}.$$

Estimates for the map $f \to qf$. If $\sqrt{\gamma}$ has s derivatives in L^p, and $\sqrt{\gamma}$ is one near infinity, then $q = \Delta\sqrt{\gamma}$ lies in the Sobolev space $W^{s,p}$ and is compactly supported.

The following theorem is reasonably well-known:

Theorem 2 *Suppose that $s < n/p$, then the bilinear map $u, v \to uv$ maps $W^{s,p} \times W^{s,p}$ into W^{s,p^*} where $1/p^* = 2/p - s/n$.*

Transposing to the dual, and looking up the definition of the space $B^*_{\cdot}$ gives:

Corollary 1 *If $q \in W^{-1/2,2n}$, then $u \to qu = m_q(u)$ satisfies*

$$\|m_q(u)\|_{B^*_1/2} \leq C\|q\|_{W^{-1/2,2n}}\|u\|_{B^*_1/2}.$$

The constant depends on the diameter of the support of q.

The estimates for G_ζ and m_q are enough to imply that the series

$$\sum_{j=0}^{\infty} (G_\zeta m_q)^j (1)$$

converges and gives exponentially growing solutions of $\Delta v - qv = 0$ if q is small in $W^{s,2n}$ and compactly supported.

A similar perturbation off of smooth potentials allows us to remove the smallness condition, at least for ζ large.

We can also show that

$$\lim_{|\zeta|\to\infty} \|\psi\|_{B^*_{1/2}} = 0.$$

The last steps–technicalities omitted. Suppose we have two conductivities γ_1 and γ_2 in $W^{3/2,p}(\Omega)$, $p > 2n$ for which the Dirichlet to Neumann maps are equal. Extend γ_j to $\mathbf{R}^n$ and construct the potentials q_j as described above.

Fix $\xi \in \mathbf{R}^n$ and R large. Because we are in $\mathbf{R}^n$, $n \geq 3$, we may choose ζ_1 and ζ_2 so that

$$\begin{aligned} \zeta_j \cdot \zeta_j &= 0, \qquad j = 1,2 \\ \zeta_1 + \zeta_2 &= -i\xi, \qquad j = 1,2 \\ |\zeta_j| &= R, \qquad j = 1,2. \end{aligned}$$

Construct solutions v_j of $\Delta v_j - q_j = 0$ of the form $v_j = e^{x\cdot\zeta_j}(1 + \psi_j)$.

Because the Dirichlet to Neumann maps are equal, we have

$$\langle (q_1 - q_2), e^{x\cdot(\zeta_1+\zeta_2)}(1+\psi_1)(1+\psi_2)\rangle = 0.$$

Let $\zeta \to \infty$ obtain that $\hat{q}_1(\xi) - \hat{q}_2(\xi) = 0$. Conclude $\gamma_1 = \gamma_2$.

Some open questions, I.

Conjecture 1 *The map*

$$\gamma \to \Lambda_\gamma$$

is injective on the set of conductivities in $W^{1,p}$, $p > n$.

This is known in two dimensions.

Some open questions, II. A simpler question is the following: It appears that the natural boundary for a uniqueness theorem for the inverse conductivity problem with conductivities in a Sobolev space is the line where the Sobolev space (almost) embeds into continuous functions. Thus, it is reasonable to conjecture the following:

Conjecture 2 *The map*

$$\gamma \to \Lambda_\gamma$$

is injective on the set of conductivities in $W^{3/2,p}$, $p > 2n/3$.

The proof of this requires better estimates for G_ζ than are known.

References.

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