Worksheet on induction MA113
Calculus I Fall 2006

First, let us explain the use of $\sum$ for summation. The notation

$$\sum_{k=1}^{n} f(k)$$

means to evaluate the function $f(k)$ at $k = 1, 2, \ldots, n$ and add up the results. In other words:

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \ldots + f(n).$$

For example:

$$\sum_{k=1}^{4} k^2 = 1 + 4 + 9 + 16,$$

$$\sum_{k=1}^{n} (2k-1) = 1 + 3 + 5 + \ldots + 2n - 1,$$

and

$$\sum_{k=3}^{2n} 1 = 2n - 2.$$

The principle of mathematical induction is used to establish the truth of a sequence of statements or formula which depend on a natural number, $n = 1, 2, \ldots$. We will use P_k to stand for a statement which depends on k. For example, P_k might stand for the statement "The number $2k - 1$ is odd." These statements are true for $k = 1, 2, \ldots$

The principle of mathematical induction is:

Principle of mathematical induction. Suppose that P_n is a sequence of statements depending on a natural number $n = 1, 2, \ldots$. If we show that:

- P_1 is true
- For $N = 1, 2, \ldots$: If P_N is true, then P_{N+1} is true.

Then, we may conclude that all the statements P_n are true for $n = 1, 2, \ldots$.

To see why this principle makes sense, suppose that we know P_1 is true, then the second step allows us to conclude P_2 is true. Now that we know P_2 is true, the second step allows us to conclude P_3 is true. If we repeat this $n - 1$ times, we conclude that P_n is true.

This principle is useful because it allows us to prove an infinite number of statements are true in just two easy steps! We usually call the first step the *base case* and the second step is called the *induction step.*

Below are several examples to illustrate how to use this principle. The statement P_N that we assume to hold is called the *induction hypothesis.* The key point in the induction step is to show how the truth of the induction hypothesis, P_N, leads to the truth of P_{N+1}.

Example 1. Show that for $n = 1, 2, 3, \ldots$, the number $n^2 - n$ is even.
Solution. Base case. This is easy. If $n = 1$, then $n^2 - n = 1^2 - 1 = 0$ and 0 is even.
Induction step. We suppose that $N^2 - N$ is even and we want to use this assumption to show that $(N+1)^2 - (N+1)$ is even. We write $(N+1)^2 - (N+1) = N^2 + 2N + 1 - N - 1 = N^2 - N + 2N$. Now $2N$ is even when N is a whole number and $N^2 - N$ is even by our induction hypothesis. As the sum of two even numbers is again even, we conclude that $(N+1)^2 + (N+1)$ is even. ∎

Example 2. Show that for $n = 1, 2, \ldots$, we have

$$\sum_{j=1}^{n} 2j = n(n+1).$$

Solution Base case. If $n = 1$, then $n(n+1) = 1 \cdot 2 = 2$. Also,

$$\sum_{j=1}^{1} 2j = 2.$$

Thus both sides are equal if $n = 1$.
Induction step. Now suppose that the formula $\sum_{j=1}^{N} 2j = N(N+1)$ is true and consider the sum

$$\sum_{j=1}^{N+1} 2j = \sum_{j=1}^{N} 2j + 2(N+1).$$

We use our induction hypothesis that $\sum_{j=1}^{N} 2j = N(N+1)$ to conclude that

$$\sum_{j=1}^{N+1} 2j = N(N+1) + 2(N+1).$$

Simplifying this last expression gives

$$N(N+1) + 2(N+1) = N^2 + N + 2N + 2 = N^2 + 3N + 2 = (N+2)(N+1).$$

Since $(N+2)(N+1) = (N+1+1)(N+1)$, we have shown that the formula

$$\sum_{j=1}^{N+1} 2j = (N+1+1)(N+1)$$

is true. This completes the induction step and thus the proof by induction. ∎

Example 3. All horses are the same color.

Solution. We will show by induction that any set of N horses consists of horses of the same color.
The base case is easy. If we have a set with one horse, then all horses in the set are the same color.
We assume as our induction hypothesis that any set of N horses consists of horses of the same color. We take a set of $N+1$ horses, and put one of the horses in the barn for a moment. By our induction hypothesis, the remaining N horses are all of the same color. Now, we put a different horse in the barn. Again, the remaining N horses are all the same color. It follows that the set of $N+1$ horses are all the same color. ∎

Below is a selection of problems related to mathematical induction. You should begin working on these problems in recitation. Write up your solutions carefully, elegantly, and in complete sentences.

1. (a) For $n = 1, 2, 3, 4$, compute
$$\sum_{k=1}^{n}(2k-1).$$
Make a guess for the value of this sum for $n = 1, 2, \ldots$.
   (b) Use mathematical induction to prove that your guess is correct.

2. Use the principle of mathematical induction to prove that
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. Define $n! = n(n-1)!$ and $0! = 1$. Thus, $n! = n(n-1)\ldots 1$, if $n = 1, 2, 3, 4, \ldots$.
   (a) Define $A_{n,k} = \frac{n!}{(n-k)!k!}$. Show that we have the formula
$$A_{n+1,k+1} = A_{n,k} + A_{n,k+1}.$$
Hint: Obtain a common denominator on the right-hand side. Do not use the principle mathematical induction to prove this.
   (b) For $n = 1, 2, \ldots$, show that there are numbers $B_{n,k}$ so that
$$(x+y)^n = \sum_{k=0}^{n} B_{n,k} x^k y^{n-k}.$$
You should be able to do this by induction on n. For the base case, write down $B_{1,0}$ and $B_{1,1}$.
In the induction step, you should write $(x+y)^{N+1} = (x+y)(x+y)^N = (x+y)(B_{N,0}x^0y^N + B_{N,1}x^1y^{N-1} + \ldots + B_{N,N}x^Ny^0)$ and discover a formula that expresses $B_{N+1,k}$ in terms of some of the numbers $B_{N,j}$, $j = 0, \ldots, N$ which are known from the induction hypothesis.

(c) Is there a relation between the numbers $A_{n,k}$ and $B_{n,k}$? (A proof is not expected.)

Additional problems. Below are some additional exercises for you to consider. You will not be able to solve all of these problems at this time. These problems will not be collected.

1. Find the flaw in the proof that all horses are the same color.

2. Let $f_1(x) = x - 2$ and then define f_n for $n = 1, 2, \ldots$ by $f_{n+1}(x) = f_1(f_n(x))$. (It is the principle of mathematical induction which tells us that these two statements suffice to define f_n for all $n = 1, 2, 3, \ldots$.) Use mathematical induction to prove that
$$f_n(x) = x - 2n.$$

3. Let P_n be the statement: $n^2 - n$ is an odd integer.

   (a) Show that if P_n is true, then P_{n+1} is true.

   (b) Is P_1 true?

   (c) Is P_n true for any n?

4. Let $f(x) = \sin(2x)$. Prove that for $n = 1, 2, \ldots$,
$$\frac{d^{2n}}{dx^{2n}} f(x) = (-4)^{2n} \sin(2x).$$

5. Prove that
$$\frac{d}{dx} x^n = n x^{n-1}, \qquad n = 1, 2 \ldots.$$
Hint: For the base case $n = 1$, use the definition of the derivative. For the induction step write $x^{n+1} = x \cdot x^n$ and use the product rule.

6. Prove that
$$\frac{d}{dx} \frac{1}{x^n} = \frac{-n}{x^{n+1}}, \qquad n = 1, 2 \ldots.$$

7. Prove that
$$\frac{d^n}{dx^n} x^n = n!, \qquad n = 0, 1, \ldots$$

8. (a) Find a simple formula for
$$\sum_{k=1}^{n} ((k+1)^2 - k^2) = 2^2 - 1 + (3^2 - 2^2) + \ldots + n^2 - (n-1)^2 + (n+1)^2 - n^2.$$

(b) Using your answer to part a), find a simple expression for

$$\sum_{k=1}^{n}(2k-1).$$

To do this you should simplify each summand on the left.

9. Use mathematical induction to prove that

$$\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

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