Worksheet on induction MA113
Calculus I Spring 2006

First, let us explain the use of the $\sum$ for summation. The notation

$$\sum_{k=1}^{n} f(k)$$

means to evaluate the function $f(k)$ at $k = 1, 2, \ldots, n$ and add up the results. In other words:

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \ldots + f(n).$$

For example:

$$\sum_{k=1}^{4} k^2 = 1 + 4 + 9 + 16,$$

$$\sum_{k=1}^{n} (2k-1) = 1 + 3 + 5 + \ldots + 2n - 1,$$

and

$$\sum_{k=3}^{2n} 1 = 2n - 2.$$

The principle of mathematical induction is used to establish the truth of a sequence of statements or formula which depend on a natural number, $n = 1, 2, \ldots$. We will use P_k to stand for a statement which depends on k. For example, P_k might stand for the statement "The number $2k - 1$ is odd." This sequence of statements is true for $k = 1, 2, \ldots$

The principle of mathematical induction is:

Principle of mathematical induction. Suppose that P_n is a sequence of statements depending on a natural number $n = 1, 2, \ldots$. If we show that:

- P_1 is true
- For $N = 1, 2, \ldots$: If P_N is true, then P_{N+1} is true.

Then, we may conclude that all the statements P_n are true for $n = 1, 2, \ldots$.

To see why this holds, suppose that we know P_1 is true, then the second step allows us to conclude P_2 is true. Now that we know P_2 is true, the second step allows us to conclude P_3 is true. If we repeat this $n - 1$ times, we know that P_n is true.

This principle is useful because it allows us to prove an infinite number of statements are true in just two easy steps! We usually call the first step the *base case* and the second step is called the *induction step.*

Below are several examples to illustrate how to use this principle. The statement P_N that we assume to hold is called the *induction hypothesis.* The key point in the induction step is to show how the truth of the induction hypothesis, P_N, leads to the truth of P_{N+1}.

Example 1. Show that for $n = 1, 2, 3, \ldots$, the number $4^n - 1$ is a multiple of 3.
Solution. Base case. We need to show this is true when $n = 1$. This is easy since $4^1 - 1 = 4 - 1 = 3$ and 3 is divisible by 3.
Induction step. We suppose that $4^N - 1$ is a multiple of 3 and we want to use this assumption to show that $4^{N+1} - 1$ is a multiple of 3. Our assumption for N means that for some whole number M, $4^N - 1 = 3M$. Now $4^{N+1} - 1$. If we add and subtract 4, we have

$$4^{N+1} - 1 = 4^{N+1} - 4 + 4 - 1 = 4(4^N - 1) + 3.$$

Now we use our induction hypothesis that $4^N - 1$ is a multiple of 3 to replace $4^N - 1$ by $3M$ and obtain that

$$4^{N+1} - 1 = 4 \cdot 3M + 3 = 3(4M + 1).$$

Thus we have shown that $4^{N+1} - 1$ is a multiple of 3.

Example 2. Show that for $n = 1, 2, \ldots$, we have

$$\sum_{j=1}^{n} 2j = n(n+1).$$

Solution Base case. If $n = 1$, then $n(n+1) = 1 \cdot 2 = 2$. Also,

$$\sum_{j=1}^{1} 2j = 2.$$

Thus both sides are equal if $n = 1$.
Induction step. Now suppose that the formula is true for N and consider the sum

$$\sum_{j=1}^{N+1} 2j = \sum_{j=1}^{N} 2j + 2(N+1).$$

We use our induction hypothesis that $\sum_{j=1}^{N} 2j = N(N+1)$ to conclude that

$$\sum_{j=1}^{N+1} 2j = N(N+1) + 2(N+2).$$

Simplifying this last expression gives

$$N(N+1) + 2(N+1) = N^2 + N + 2N + 2 = N^2 + 3N + 2 = (N+2)(N+1).$$

Since $(N+2)(N+1) = (N+1+1)(N+1)$, we have shown that the formula

$$\sum_{j=1}^{N+1} 2j = (N+1+1)(N+1)$$

is true. This completes the proof by induction.

Below is a selection of problems related to mathematical induction. You should begin working on these problems in recitation. Write up your solutions carefully, elegantly, and in complete sentences.

1. (a) For $n = 1, 2, 3, 4$, compute
   $$\sum_{k=1}^{n} (2k-1).$$
   Make a guess for the value of this sum for $n = 1, 2, \ldots$.

   (b) Use mathematical induction to prove that your guess is correct.

2. Use mathematical induction to prove that
   $$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6.$$

3. Let $f_1(x) = x - 2$ and then define f_n for $n = 1, 2, \ldots$ by $f_{n+1}(x) = f_1(f_n(x))$. (It is the principle of mathematical induction which tells us that these two statements suffice to define f_n for all n.) Use mathematical induction to prove that
   $$f_n(x) = x - 2n.$$

4. Let P_n be the statement: $n^2 - n$ is an odd integer.

   (a) Show that if P_n is true, then P_{n+1} is true.

   (b) Is P_1 true?

   (c) Is P_n true for any n?

Below are some additional exercises. You may not be able to solve all of these problems at this time. These problems will not be collected.

1. Let $f(x) = \sin(2x)$. Prove that for $n = 1, 2, \ldots$,
   $$\frac{d^{2n}}{dx^{2n}} f(x) = (-4)^{2n} \sin(2x).$$

2. Prove that
$$\frac{d}{dx}x^n = nx^{n-1}, \qquad n = 1, 2 \ldots.$$
Hint: For the base case $n = 1$, use the definition of the derivative. For the induction step write $x^{n+1} = x \cdot x^n$ and use the product rule.

3. Prove that
$$\frac{d}{dx}\frac{1}{x^n} = \frac{-n}{x^{n+1}}, \qquad n = 1, 2 \ldots.$$

4. Prove that
$$\frac{d^n}{dx^n}x^n = n!, \qquad n = 0, 1, \ldots$$

5. (a) Find a simple formula for
$$\sum_{k=1}^{n}((k+1)^2 - k^2) = 2^2 - 1 + (3^2 - 2^2) + \ldots + n^2 - (n-1)^2 + (n+1)^2 - n^2.$$

   (b) Using your answer to part a), find a simple expression for
$$\sum_{k=1}^{n}(2k-1).$$
   To do this you should simplify each summand on the left.

6. Use mathematical induction to prove that
$$\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

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