Before beginning, it might be helpful to recall the quadratic formula. The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity inside the radical, $b^2 - 4ac$, is called the *discriminant.* It is easy to see that we have two real roots if the discriminant is positive, one real root if the discriminant is 0 and no real roots if the discriminant is negative.

1. Find the slope of the tangent line to the graph of $f(x) = x^2$ at a general point $x = a$. Use the definition.

2. If we consider the parabola $y = x^2$, and a point (c, d) in the plane, how many tangent lines to the parabola are there that pass through (c, d) (which may not lie on the parabola)? The exercises below answer this question and allow you to relate the number of tangent lines to the location of the point.

   (a) Make three sketches which show the tangent line(s) to $y = x^2$ which pass through

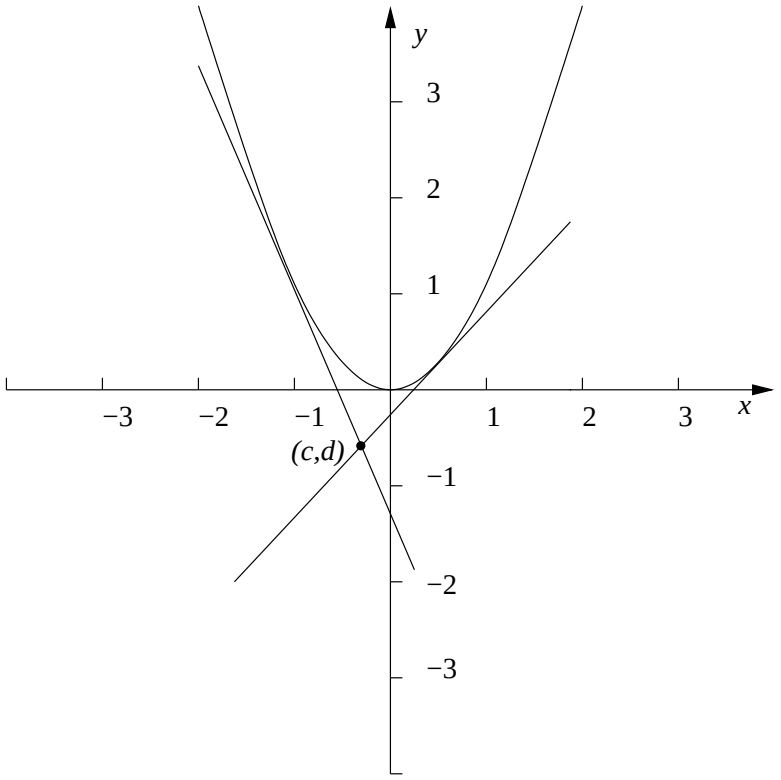
      i. $(1, -2)$
      ii. $(1, 1)$
      iii. $(0, 1)$.

   (b) Make a conjecture as to how many tangent lines of the parabola pass through a given point (c, d). How does the answer depend on the point (c, d)?

   (c) Write the equation of the tangent line to the parabola $y = x^2$ at (a, a^2).

   (d) If we require the tangent line in part c) to pass through point (c, d), we obtain an equation for a, the x-coordinate of the point where the tangent line meets the parabola. Write out this equation.

   (e) Tell how many solutions the equation you found in part d) has. How does the number of solutions depend on (c, d)? The best answers to this question will refer to the discriminant of a quadratic equation. Can you interpret your answer geometrically?



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