(1) Let n be a positive integer. Show that $\sum_{k=1}^{n} (4k+1) = n(2n+3)$.

Proof: If $n=1$, the left-hand side becomes $4 \cdot 1 + 1 = 5$

and the right-hand side becomes $1 \cdot (2 \cdot 1 + 3) = 5$. Both are equal,

thus the claim is true for $n=1$.

For the induction step, assume that we know the claim for some $n \geq 1$. We have

to show:

$$\sum_{k=1}^{n+1} (4k+1) = (n+1)\left[2(n+1)+3\right]. \qquad (*)$$

Using the induction hypothesis, we get for the left-hand side

$$\sum_{k=1}^{n+1} (4k+1) = \sum_{k=1}^{n} (4k+1) + \left[4(n+1)+1\right] = n(2n+3) + \left[4n+5\right] = 2n^2 + 7n + 5.$$

Working out the right-hand side of $(*)$ we get:

$$(n+1)\left[2(n+1)+3\right] = (n+1)(2n+5) = 2n^2 + 7n + 5.$$

Hence both sides of $(*)$ are indeed equal. ∎

(2) Show that the equation $x^5 + x^2 = 42 - x$ has a root.

Proof: The equation has a root iff the function $f(x) := x^5 + x^2 + x - 42$ has an

x-intercept. Since $f(1) = 3 - 42 < 0$ and $f(3) = 243 + 9 + 3 - 42 > 0$ and

f is continuous on $[1,3]$ (f is a polynomial function), the Intermediate

Value Theorem guarantees the existence of some $c \in (1,3)$ such that $f(c) = 0$.

This number c is a root of the given equation.

(3) Find all numbers c such that the function

$$f(x) = \begin{cases} 4x-1 & \text{if } x \geq 1 \\ c^2x^2+2cx & \text{if } x < 1 \end{cases}$$

is continuous and differentiable, respectively.

Sol.: As polynomial functions the pieces of f are differentiable. Thus it remains to investigate f at $x=1$. We use one-sided limits and compute:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4x-1) = 4\cdot 1 - 1 = 3 = f(1)$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (c^2x^2+2cx) = c^2+2c.$$

Hence f is continuous at 1 iff $c^2+2c=3$ iff $0 = c^2+2c-3 = (c+3)(c-1)$

iff $c=1$ or $c=-3$.

Since f is not diff. if f is not cont., we can focus on $c=1$ and $c=-3$ for checking differentiability. In both cases we get

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{4x-1-3}{x-1} = \lim_{x\to 1^+} \frac{4(x-1)}{x-1} = 4.$$

For $c=1$ we obtain

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{x^2+2x-3}{x-1} = \lim_{x\to 1^-} \frac{(x+3)(x-1)}{x-1} = \lim_{x\to 1} (x+3) = 4.$$

Hence f is diff. at $x=1$ and $f'(1)=4$.

For $c=-3$, we get

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{9x^2-6x-3}{x-1} = \frac{d}{dx}\left(9x^2-6x\right)\Big|_{x=1} = 18x-6\Big|_{x=1} = 12 \neq 4,$$

thus f is not diff. at $x=1$ in this case.

Hence we have shown: f is continuous on $\mathbb{R}$ iff $c=1$ or $c=-3$.

f is differentiable on $\mathbb{R}$ iff $c=1$.

④ Find all numbers c such that $\lim_{x\to 4} \dfrac{x^2-cx+4}{x-4}$ exists.

Sol.: Since $\lim_{x\to 4}(x-4)=0$, the limit can only exist if

$$0=\lim_{x\to 4}(x^2-cx+4) = 16-4c+4 = 20-4c, \text{ thus } c=5.$$

If $c=5$, the limit does indeed exist because

$$\lim_{x\to 4}\frac{x^2-5x+4}{x-4} = \lim_{x\to 4}\frac{(x-4)(x-1)}{x-4} = \lim_{x\to 4}(x-1) = 3.$$

Hence, the limit does exist iff $c=5$.

⑤ Given that the tangent to $y=f(x)$ at 4 is $y=3x-2$, find the equation of the tangent to the graph of $h(x) := x\cdot f(x)$ at $x=4$.

Sol.: Since the tangent to f at $x=4$ passes through $(4, f(4))$, we get $f(4)=3\cdot 4-2=10$.

The slope of this tangent is $f'(4)=3$. Thus, we get $h(4)=4\cdot f(4)=4\cdot 10=\underline{40}$

and, using the product rule, $h'(x)=f(x)+x\cdot f'(x)$, thus

$$h'(4) = 10+4\cdot 3 = \underline{22}.$$

We conclude that the equation of the tangent to $y=h(x)$ at $x=4$ is

$y-h(4)=h'(4)(x-4)$, i.e. $y-40=22(x-4)$ or $\underline{\underline{y=22x-48}}$.

⑥ Find the derivative of $f(x)=\dfrac{5x^2+3}{x^3+4}$

Sol. The quotient rule provides:

$$f'(x)=\frac{5\cdot 2x\,(x^3+4)-(5x^2+3)\cdot 3x^2}{(x^3+4)^2} = \frac{10x^4+40x-15x^4-9x^2}{(x^3+4)^2} = \underline{\underline{\frac{-5x^4-9x^2+40x}{(x^3+4)^2}}}.$$