1. Carry out the following steps to sketch the graph of

$$f(x) = \frac{x}{1+x^2}.$$

(a) Find the local maxima and minima for f. Compute the local maximum and minimum values. Give the intervals of increase and decrease.

(b) Find the inflection points for f. Give the intervals where f is concave up and concave down.

(c) Determine if f is even or odd.

(d) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

(e) Make a careful sketch of the graph of f that reflects the above information.

Solution. a) The derivative of $f(x)$ is

$$f'(x) = \frac{1-x^2}{(1+x^2)^2}.$$

We have $f'(x) = 0$ if $x = +1$ or -1. Testing values gives:

	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Test point	-2	0	2
$f'(x)$	$-3/25$	1	$-3/25$
	decreasing	increasing	decreasing

Using the first derivative test for increasing or decreasing functions, we see that f is decreasing on the intervals $(-\infty, -1)$ and $(1, \infty)$ and that f is increasing on the interval $(-1, 1)$.

Using the first derivative test for local extreme values, we see that f has a local minimum at $x = -1$ and $f(-1) = -1/2$ is the only local minimum value. Also, f has a local maximum at $x = 1$ and $f(1) = 1/2$ is a local maximum value.

(Some papers may use the second derivative test to classify local extrema. Give credit. Encourage students to learn how to use the first and second derivative to classify local extrema.)

b) We compute the second derivative,

$$f''(x) = \frac{2x(x^2-3)}{(1+x^2)^3}.$$

The solutions of $f''(x) = 0$ are $x = 0, \pm\sqrt{3}$. We make a table to determine the sign of f'' and hence the intervals where f is concave up or down.

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
Test point	-2	-1	1	2
$f''(x)$	$-4/125$	$1/2$	$-1/2$	$4/125$
concave	down	up	down	up

The second derivative test for concavity tells that f is concave up on the intervals $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$. The second derivative test for concavity tells that f is concave down on the intervals $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.

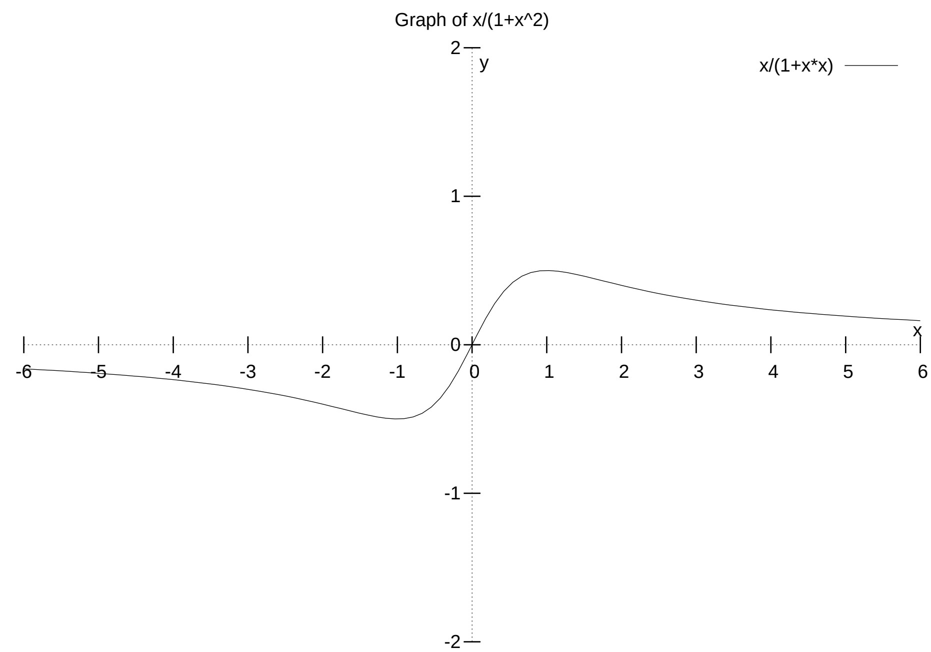
From the table, we see that f changes concavity at $x = 0, \sqrt{3}, -\sqrt{3}$, so the inflection points are $(-\sqrt{3}, -\sqrt{3}/4)$, $(0, 0)$ and $(\sqrt{3}, \sqrt{3}/4)$.

c) The function f is odd since we have that $f(-x) = -x/(1 + x^2) = -f(x)$.

d) If we take the limit, we find

$$\lim_{x\to\infty} \frac{x}{1+x^2} = \lim_{x\to-\infty} \frac{x}{1+x^2} = 0.$$

e) The graph may be found below.



∎

2. Let A be a real number and consider the cubic polynomial, $f(x) = x^3 + Ax^2 + 3x$.

   (a) Find $f'(x)$.

   Solution. The derivative is $f'(x) = 3x^2 + 2Ax + 3$. ∎

(b) Find the value(s) of A for which f has exactly one critical number.

Solution. Since f is a polynomial and differentiable for all real numbers, to find the critical numbers of f, we need to solve $f'(x) = 0$ or $3x^2 + 2Ax + 3 = 0$. The solutions are

$$x = \frac{-2A \pm \sqrt{4A^2 - 36}}{6}.$$

Thus, we will have one critical number if the discriminant, $4A^2 - 36 = 0$ or if A is in $\{3, -3\}$. ∎

(c) Find the value(s) of A for which f has two critical numbers.

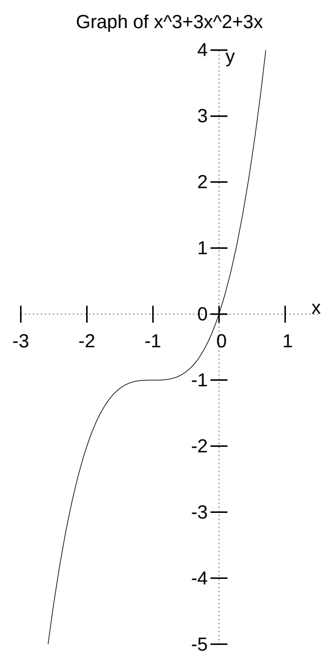
Solution. We will have two critical numbers if the discriminant, $4A^2 - 36 > 0$ or if $A^2 > 9$. The solution set of this inequality is $(-\infty, -3) \cup (3, \infty)$, or in set notation $\{A : A > 3 \text{ or } A < -3\}$. ∎

(d) Find the value(s) of A for which f has no critical numbers.

Solution. We will have no critical numbers[1] if the discriminant, $4A^2 - 36 < 0$ or if $A^2 < 9$. The solution set of this inequality is $(-3, 3)$, or $\{A : -3 < A < 3\}$. ∎

(e) Draw a sketch of the graph of the polynomial when $A = 3$ and verify that your sketch agrees with your answers to (b-d).

When $A = 3$, we are in the case discussed in part b). There is one critical number.



[1] We do not consider complex solutions in this course.

Hint: For this problem, it may be helpful to recall that a quadratic equation, $ax^2 + bx + c = 0$ will have two distinct roots if the *discriminant* $b^2 - 4ac > 0$, one root if the discriminant is 0 and no real roots if the discriminant is negative.

Grading guidelines: 1. a) Intervals (1 point), extremes (1 point), b) intervals (1 point), inflection points (1 point), c) answer (1 point), d) limits (1 point), e) sketch (1 point).
2. b-d) Critical numbers (1 point), cases (1 point). e) sketch (1 point).
Deductions: Unlabeled axes. Lack of explanation. Not using complete sentences. Mark all offences. Deduct 1 point for two offences.

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