1 Lecture 01: Functions and inverse functions

- Functions, domain, and range.
- Composition
- Graphs of lines and parabolas

The function is a central idea in mathematics. Given two sets D and Y, a *function* f takes an element of the set D and returns a value in the set Y. We usually write $f(x)$ for the value of f at x. The set D is called the *domain* of the function. Inside the set Y is the *range* of f, the set $R = \{y \in Y : y = f(x) \text{ for some } x \in D\}$.

In this course, we will usually consider functions whose domain is the real number line $\mathbf{R}$ or a union of intervals in the real line. We may specify the domain of the function as part of the definition, in some problems physical constraints will limit the domain, otherwise the domain is the largest set of real numbers where the formula is defined. (Some books call this the natural domain of the function; ours does not provide a name for this object.)

Example. Give the domain and range of the functions

$$f(x) = 2 - \sqrt{x+3}, \qquad g(x) = 2 + \frac{1}{x+1}.$$

Give your answer as a set and using interval notation.

Solution. The square root function $\sqrt{t}$ is defined for $t \geq 0$. Thus $\sqrt{x+3}$ will be defined for $x + 3 \geq 0$ or $x \geq -3$. The domain of f is $(-3, \infty]$ or $\{x : x \geq -3\}$. To find the range, note that $\sqrt{t}$ is always positive or zero. Thus $2 - \sqrt{x+3}$ will take values which are 2 or less. The range of f is

$$(-\infty, 2] \qquad \text{or} \qquad \{x : x \leq 2\}.$$

For the function g, a problem arises when we try to divide by zero. The formula defining g makes sense if $x + 1 \neq 0$ or $x \neq -1$. Thus domain of g is

$$(-\infty, -1) \cup (-1, \infty) \qquad \text{or} \qquad \{x : x \neq -1\}.$$

To find the range, we can ask if we can solve the equation $y = 2 + 1/(x+1)$. Solving we find

$$\begin{aligned} y &= 2 + \frac{1}{x+1} && \text{subtract 2 from each side} \\ y - 2 &= \frac{1}{x+1} && \text{multiply by } x+1 \text{ and divide by } y-2 \\ x + 1 &= \frac{1}{y-2} \\ x &= -1 + \frac{1}{y-2} \end{aligned}$$

Thus, we see that we can find a value x with $g(x) = y$ when the above steps are right. The one place a problem might arise is when we divide by $y - 2$. We cannot divide by zero so we must have $y \neq 2$. Thus the range is

$$(-\infty, 2) \cup (2, \infty) \qquad \text{or} \qquad \{y : y \neq 2\}.$$

∎

1.1 Composition

If we have two functions $f : X \to Y$ and $g : Y \to Z$, then we use $g \circ f$ to denote the *composition* of f and g. This is function defined by $g \circ f(x) = g(f(x))$.

Example. If $f(x) = \sqrt{x}$ and $g(x) = x + 2$ find the composite function $f \circ g$ and give the domain.

Solution. The composite function is $f(g(x)) = \sqrt{x+2}$ which is defined for $x \geq -2$ or the interval $[-2, \infty)$. ∎

1.2 Inverse functions

We say that g is the *inverse* of the function f if $f(g(x)) = x$ for a all x in the domain of f and $g(f(x)) = x$ for all x in the domain g. It is clear that if g is the inverse of f, then f is in the inverse function for g¿ We usually use the symbol f^{-1} to stand for the inverse function. If f^{-1} exists, we say that f is *invertible.* Be careful, in general f^{-1} does not mean the reciprocal $1/f$.

If a function f has an inverse, then the value of x that solves the the equation $f(x) = y$ must be $x = g(f(x)) = g(y)$. There is only one solution to the equation $f(x) = y$. Thus this function is *one-to-one.* This means that for each y, there is at most one solution to the equation $f(x) = y$.

A function is one-to-one if and only if it has an inverse. If (x, y) is a point on the graph of f, then (y, x) will be a point on the graph of f^{-1}. The picture in figure 1.2 below illustrates this.

Example. Let f be the function $f(x) = 1 + x^2$. Is f invertible?

Let $g(x) = 1 + x^2$ with the domain $[0, \infty)$ and give the graph of the inverse function and the domain and range.

Solution. The function f is not invertible. For example, the equation $f(x) = 5$ has two solutions, $x = 2$ and -2 and the function does not have an inverse.

If we restrict to x in $[0, \infty)$, then the equation $x^2 + 1 = y$ has one solution when $y \geq 1$ and zero solutions if $y < 0$. The solution is $\sqrt{y - 1}$. Thus the inverse function g^{-1} exists and function, $g^{-1}(x) = \sqrt{x - 1}$. The domain of g is $[0, \infty)$ and the range is $[1, \infty)$. Thus the domain of g^{-1} is $[1, \infty)$ and the range is $[0, \infty)$. The graphs of g and g^{-1} are in figure 1.2. ∎

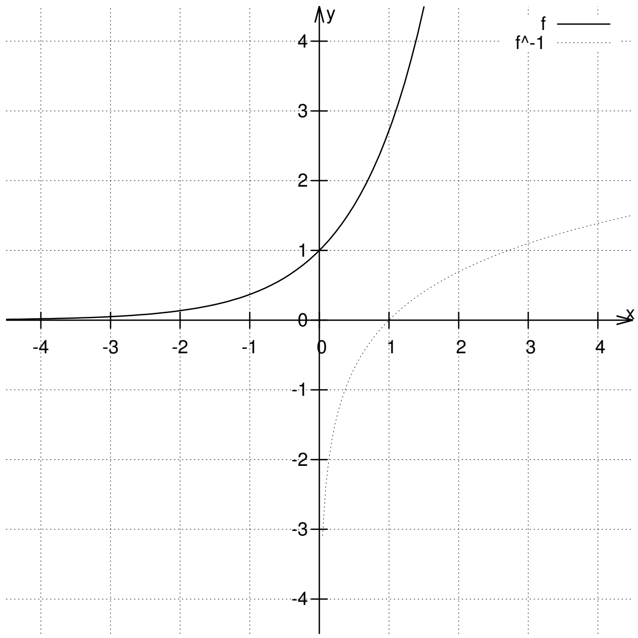

Figure 1: Graphs of f and f^{-1}.

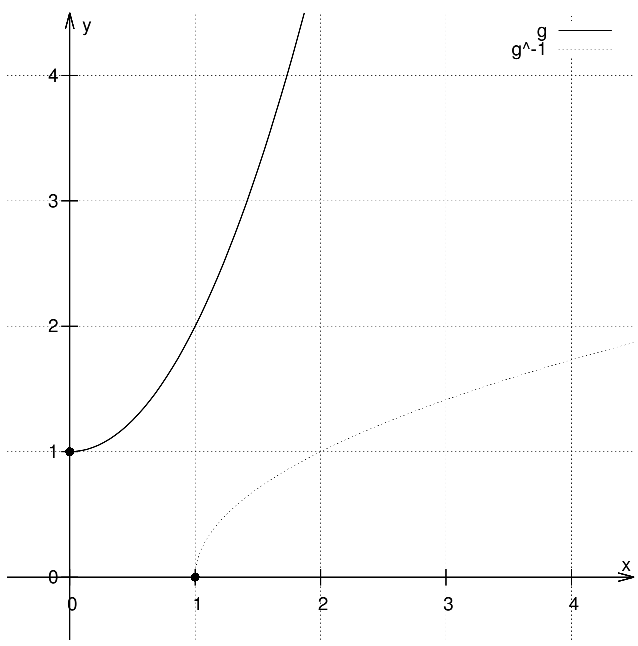

Figure 2: Graphs of $g(x) = 1 + x^2$ with domain $[0, \infty)$ and g^{-1}.

Example. Find the inverse of $f(x) = (x + 1)/(x - 2)$. Give domain and range.

Solution. To find a formula for the inverse, we need to solve the equation

$$y = \frac{x+1}{x-2}.$$

The solution is

$$x = \frac{1+2y}{y-1}$$

unless $y = 1$. Thus, the inverse function $f^{-1}(x) = \frac{1-2x}{1-x}$.

The domain of f is $(-\infty, 2) \cup (2, \infty)$ and the range is $(-\infty, 1) \cup (1, \infty)$.

The domain of f^{-1} is $(-\infty, 1) \cup (1, \infty)$ and the range is $(-\infty, 2) \cup (2, \infty)$. ∎

January 15, 2014