1 Lecture 02: Review of trig

- Definition of sin and cos.
- Definition of remaining trig functions
- Pythagorean identities, addition formulae

1.1 Definitions

If an angle in a circle of radius r cuts off an arc of length s, then the radian measure of the angle is s/r.

Example. Give the radian measure of a full circle? a right angle?

Solution. Since a circle of radius r has circumference $2\pi r$, it follows that the radian measure of the full circle is 2π.

For a right angle, we note that a circle contains 4 right angles, so the radian measure of a right angle is $2\pi/4 = \pi/2$.

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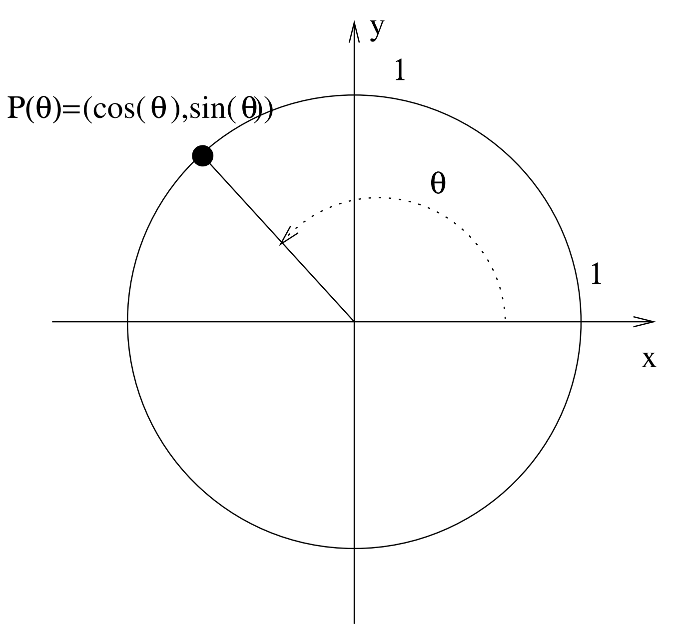
Does the measure of an angle depend on the circle we use? Why? No–if we change the radius, each length is multiplied by the same factor.

To define the basic trig functions $\cos(\theta)$ and $\sin(\theta)$, we draw a unit circle, move anti-clockwise to form an angle of measure θ and let $P(\theta)$ denote the point where the terminal side crosses the circle. The coordinates of this point give $(\cos(\theta), \sin(\theta))$,

$$P(\theta) = (\cos(\theta), \sin(\theta)).$$

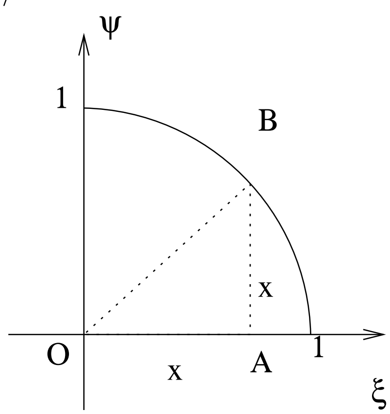


Example. Find $\cos(\theta)$ and $\sin(\theta)$ for the angles $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$.

Solution. We begin with the angle $\pi/4$. If we draw $P(\theta)$, we see that the triangle OAB in the figure below has a right angle at A and an angle of $\pi/4$ at). Since the angles sum to $\pi/2$, the angle with vertex at B must be $\pi/4$ also. Thus OAB is an isosceles right triangle with hypotenuse of length 1. If we let x denote the common length of the two sides, from the Pythagorean theorem we obtain

$$x^2 + x^2 = 1$$

or $x^2 = 1/2$. Solving for x gives $x = 1/\sqrt{2} = \sqrt{2}/2$. Thus $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$.



For the angle $3\pi/4$, we obtain a point in the second quadrant with $\cos(\pi/4) < 0$ and $\sin(\pi/4) > 0$. Thus $\cos(\pi/4) = -\sqrt{2}/2$ and $\sin(\pi/4) = \sqrt{2}/2$.

In the third quadrant, both are negative thus $\cos(5\pi/4) = \sin(5\pi/4) = -\sqrt{2}/2$. In the fourth quadrant, $\cos(\pi/4) = \sqrt{2}/2 > 0$ and $\sin(\pi/4) = -\sqrt{2}/2 < 0$. ∎

Finally, we recall the remaining trig functions. These all are defined in terms of sin and cos.

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)}.$$

Example. Give the domain of the function $\sec(x)$.

Solution. From examining the unit circle we see that $\cos(x)$ is defined on $(-\infty, \infty)$, but is zero at $\pi/2$ and $\pi/2$ plus any multiple of π. Thus the domain of the secant function is

$$\{x : x \neq \frac{\pi}{2} + k\pi, k = 0, \pm 1, \pm 2, \ldots\}.$$

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Example. Work out the values of all of the trigonometric functions at the angles $\pi/6$ and $\pi/3$.

1.2 Identities

We recall several important identities for the trig functions. First, since the point $P(\theta)$ lies on the unit circle, we have

$$\cos^2(\theta) + \sin^2(\theta) = 1. \tag{1}$$

Next, if we divide this identity by $\cos^2(\theta)$, we obtain

$$1 + \tan^2(\theta) = \sec^2(\theta).$$

Dividing (1) by $\sin^2(\theta)$ gives

$$1 + \cot^2(\theta) = \csc^2(\theta).$$

If we let $P(\theta) = (x, y)$, then we can see that $P(-\theta) = (x, -y)$, it follows that sin is odd and cos is even

$$\cos(-\theta) = \cos(\theta), \qquad \sin(-\theta) = -\sin(\theta).$$

Finally, we recall the addition formula for sine and cosine.

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

and

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

If we set $x = y$, we obtain the double angle formulae

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

and using the Pythagorean identity (1) we obtain

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta).$$

Example. Observe that $\sin(\pi/3) = \cos(\pi/6)$ and $\cos(\pi/6) = \sin(\pi/3)$. Use these observations and the double-angle formula for cos to find $\cos(\pi/3)$.

Solution. Using that the angles $\pi/3$ and $\pi/6$ are complementary, we see that $\cos(\pi/3) = \sin(\pi/6)$. From the double angle formula

$$\cos(\pi/3) = 1 - 2\sin^2(\pi/6).$$

and substituting for $\sin(\pi/6)$ gives

$$\cos(\pi/3 = 1 - 2\cos^2(\pi/3).$$

Rearranging gives the quadratic equation

$$2\cos^2(\pi/3) + \cos(\pi/3) - 1 = 0.$$

Solving this quadratic equation for $\cos(\pi/3)$ gives

$$\cos(\pi/3) = \frac{-1 \pm \sqrt{1 + 8}}{4} = 1/2 \text{ or } -1.$$

Since $\cos(\pi/3) > 0$, we have $\cos(\pi/3) = 1/2$.

$\blacksquare$

1.3 Inverse trigonometric functions

None of the trigonometric functions are one-to-one. However, by restricting the domain we obtain a one-to-one function. The standard choices for domains are below:

$$\begin{array}{ll} \sin(x) & [-\pi/2, \pi/2] \\ \cos(x) & [0, \pi] \\ \tan(x) & (-\pi/2, \pi/2) \\ \sec(x) & [0, \pi/2) \cup (\pi/2, \pi] \end{array}$$

The inverse function to sin on the domain $[-\pi/2, \pi/2]$ will be denoted using either the notation $\sin^{-1}$ or arcsin. The prefix arc suggests that when we find $\theta = \arcsin(x)$, we are looking for the angle or arc which has $\sin(\theta) = x$. Similar considerations apply to arccos or $\cos^{-1}$, arctan or $\tan^{-1}$, and arcsec or $\sec^{-1}$.

Note that there is an inconsistency in our use of the notation $\sin^{-1}$. The sin function does not have an inverse. Rather, we are taking the inverse of the function g with $g(x) = \sin(x)$ for x in the domain $[-\pi/2, \pi/2]$. The notation $\sin^{-1}$ is ambiguous because it is not clear if it represents the inverse function arcsin or the reciprocal $1/sin$ = csc. Perhaps the next time we invent mathematics, we can find better notation.

Example. Sketch the graph of $\arcsin(x)$. Give the domain and range. Find $\arcsin(1/2)$.

Solution. We sketch the graph of $y = \sin(x)$ for x in $[-\pi/2, \pi/2]$. Several convenient points on the graph include $(-\pi/2, -1)$, $(0, 0)$, and $(\pi/2, 1)$. Thus the points $(-1, -\pi/2)$, $(0, 0)$, and $(1, \pi/2)$ lie on the graph of $\sin^{-1}$ or arcsin. Plotting these points and doing our best to fill in the intermediate points gives the graph in Figure 1.3.

We are considering $\sin(x)$ with domain $[-\pi/2, \pi/2]$ and range $[-1, 1]$. The inverse function arcsin will have domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.

To find $\arcsin(1/2)$, we recall that in a $\pi/6$, $\pi/3$, $\pi/2$ triangle, with hypotenuse 1, the legs are of length $1/2$, $\sqrt{3}/2$ and $\sin(\pi/6) = 1/2$. Thus $\arcsin(1/2) = \pi/6$.

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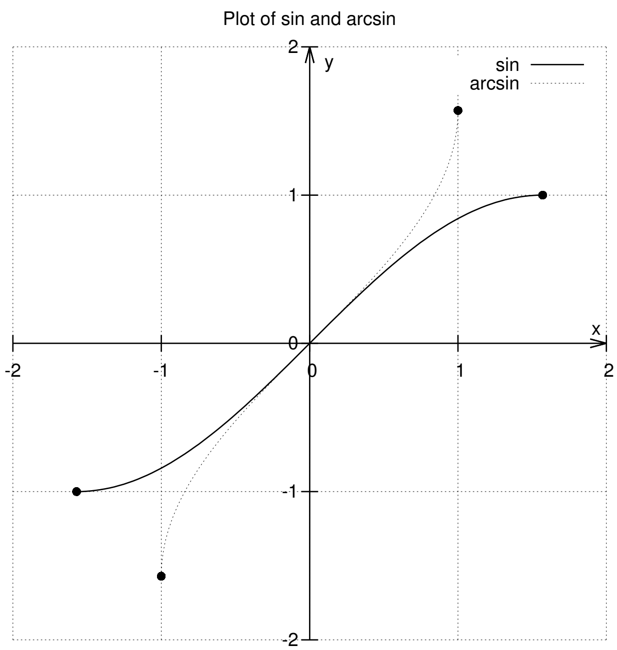

Figure 1: Graph of $\sin(x)$ on $[-\pi/2, \pi/2]$ and $\sin^{-1}(x)$