1 Lecture 05: The limit laws

- Some basic limits and the limit laws
- Examples: Using the limit laws to compute limits. Limits of polynomials
- Examples where the limit laws do not apply

1.1 Limit laws

In this section, we will outline how we can systematically compute limits. This relies on two steps: 1) we will give a couple of elementary limits 2) we will give laws or rules that help us to combine the elementary limits to study more interesting functions. We will not give a careful proof of the limit laws. To give a proof involves providing a more careful definition of limits than we have given in this course.

Theorem 1 *Suppose f and g are functions defined in an open interval containing a, except perhaps at a and we have*

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M.$$

Then

$$\lim_{x\to a}(f(x)+g(x)) = L+M \tag{2}$$

$$\lim_{x\to a}(f(x)g(x)) = L\cdot M \tag{3}$$

$$\lim_{x\to a}\frac{1}{g(x)} = \frac{1}{M}, \qquad \textit{if } M\neq 0. \tag{4}$$

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{L}, \qquad \textit{if } n=2,3,\ldots \textit{ and } L>0 \textit{ if } n \textit{ is even.} \tag{5}$$

We emphasize that in this theorm L and M are not allowed to be infinite. Arithmetic with infinity can be very tricky. For example $\lim_{x\to 0} ax^2 = 0$, $\lim_{x\to 0}\frac{1}{x^2} = +\infty$, and $\lim_{x\to 0}(ax^2\frac{1}{x^2}) = a$. Thus there is no way to define $0\cdot\infty$ so that the limit law (3) holds.

Our basic limits are.

Proposition 6 *If a and c are real numbers, then*

$$\lim_{x\to a} c = c, \qquad \lim_{x\to a} x = a.$$

Using the limit laws, we can evaluate large classes of limits. For example, we may take the limit of any polynomial.

Example. Find the limit of $\lim_{x\to 3} 5x^2 - 2$.

Solution. We may use (2) and (3) to obtain

$$\lim_{x\to 3}(5x^2-2) = (\lim_{x\to 3} 5)(\lim_{x\to 3} x)(\lim_{x\to 3} x) + \lim_{x\to 3}(-2).$$

Each of the limits on the right is given in Proposition 6 Thus, we have

$$\lim_{x\to 3}(5x^2-2) = 5\cdot 3^2 + -2 = 43.$$

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It should be easy to see that by following the arguments above we can show that if R is a rational function and a is in the domain of R,

$$\lim_{x\to a} R(x) = R(a).$$

Also, for a rational number p/q,

$$\lim_{x\to a} x^{p/q} = a^{p/q}$$

provided $a > 0$ if q is even and $a \neq 0$ if $p/q < 0$.

1.2 Limits where the limit rules do not apply

Some more interesting examples include

$$\lim_{x\to 1}\frac{x^2-1}{x-1}, \qquad \lim_{x\to 1}\frac{x+1}{x^2-1} \qquad \lim_{x\to 0}\frac{\sin(x)}{x}.$$

In each of these limits, the limit rules do not apply. We are tempted to use the quotient rule, but the limit of the denominator in each fraction is 0. The first two may be studied by simplifying. We are not yet ready to study the third.

Example. Find the limits

$$\lim_{x\to 1}\frac{x^2-1}{x-1}, \qquad \lim_{x\to 1}\frac{x+1}{x^2-1}$$

Solution. We cannot directly apply the rule for the limit of a quotient to evaluate the first limit since the limit of the denominator, $\lim_{x\to 1}(x-1) = 0$. However, this does not mean the limit does not exist. We simplify

$$\frac{x^2-1}{x-1} = x+1.$$

Since this is valid for all $x \neq 1$, we may simplify both sides to obtain

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1}(x + 1) = 2.$$

For the second, we simplify

$$\frac{x + 1}{x^2 - 1} = \frac{1}{x - 1}.$$

We have

$$\lim_{x \to 1^-} \frac{1}{x - 1} = -\infty \qquad \lim_{x \to 1^+} \frac{1}{x - 1} = +\infty.$$

Thus the limit $\lim_{x \to 1} \frac{x+1}{x^2-1}$ does not exist. ∎

Example. Suppose that the position function of a particle is given by $p(t) = t^2$. Find the instantaneous velocity at a.

Solution. On a small interval $[a, t]$, the average velocity is

$$\frac{p(t) - p(a)}{t - a} = \frac{t^2 - a^2}{t - a} = t + a.$$

Taking the limit,

$$\lim_{t \to a} \frac{t^2 - a^2}{t - a} = \lim_{t \to a}(t + a) = 2a.$$

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January 26, 2014