1 Lecture 07: Algebraic computation of limits

- Algebraic methods for evaluating limits.
- Applications. Find a tangent line.

1.1 Algebraic methods

We have discussed various ways of finding limits including numerical estimation, studying a graph, and using the limit laws. In the end, the limit laws provide a careful proof. To evaluate a limit at a requires that we consider *all* values near a. This can take a long time on a calculator. A graph may not provide enough detail if the function has interesting features that are very small. If we accept the limit laws, they provide a path to a careful evaluation of a limit. In our previous lectures we have run into a number of limits where we cannot initially apply our limit laws, but after an algebraic simplification, we are able to apply these laws. Today we will look at several such limits and also try to explain where they arise in nature. Your textbook contains a number of additional examples.

Example. Find the limit

$$\lim_{x \to 0} \frac{\tan(x)}{\sin(x)}.$$

Solution. We recall the rule for the limit of a quotient–if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M \neq 0$, then $\lim_{x\to a} f(x)/g(x) = L/M$. This does not apply directly since $\lim_{x\to 0} \sin(x) = 0$. Since $\sin(x)$ is continuous everywhere, this follows by the direct substitution rule.

However, we can simplify and obtain

$$\frac{\tan(x)}{\sin(x)} = \frac{1}{\sin(x)} \frac{\sin(x)}{\cos(x)} = \frac{1}{\cos(x)},$$

at least when $\sin(x) \neq 0$. Since $\tan(x)/\sin(x) = 1/\cos(x)$ when x is not a multiple of π, we have

$$\lim_{x\to 0} \frac{\tan(x)}{\sin(x)} = \lim_{x\to 0} \frac{1}{\cos(x)} = 1.$$

by direct substitution rule since $\cos(x) \neq 0$. ∎

Here is another example using the limit laws in a different way.

Example. Suppose that $\lim_{h\to 5} hf(h) = 4$, find $\lim_{h\to 5} f(h)$.

Solution. We write $f(h) = hf(h)/h$ and use the rule for the limit of a product,

$$\lim_{h\to 5} f(h) = \lim_{h\to 5}(hf(h)/h) = \lim_{h\to 5}(1/h) \cdot \lim_{h\to 5} hf(h) = 4/5.$$

∎

We give an extended example that includes a reminder of how limits are used in finding tangent lines.

Example. Find the slope of the secant line to the graph of $y = \sqrt{x}$ that passes through (4,2) and $(x, \sqrt{x})$. Does the limit of this slope exist as x approaches 2?

Find the tangent line to the graph of $y = \sqrt{x}$ at the point $(x, y) = (4, 2)$.

Solution. We recall that to find the tangent line at 4, we are to consider the secant line through $(4, 2)$ and $(x, \sqrt{x})$. We compute the slope

$$\frac{\sqrt{x} - 2}{x - 4}.$$

We use the trick of multiplying by the conjugate to simplify,

$$\frac{\sqrt{x} - 2}{x - 4}\frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \frac{1}{\sqrt{x} + 2}\frac{x - 4}{x - 4}.$$

And since $1/(\sqrt{x} + 2)$ is continuous at $x = 4$, we may use the direct substitution rule to find

$$\lim_{x\to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}.$$

Finally, recall the tangent line must pass through $(4, 2)$ and have slope $1/4$, thus its equation is

$$y - 2 = \frac{1}{4}(x - 4).$$

We may check our solution by plotting $\sqrt{x}$ and the tangent line on the same axes (see Figure 1.1).

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You should work a selection of problems and make sure you are familiar with the algebraic techniques necessary. Here are a few for practice.

Exercise. Find the limits

$$a) \lim_{x\to -2} \frac{x^3 + 8}{x - 2}, \qquad b) \lim_{x\to 3}(\frac{1}{x - 3} - \frac{6}{x^2 - 9}), \qquad c) \lim_{h\to 0}(\frac{1}{h}(\frac{1}{x + h} - \frac{1}{x})).$$

Exercise. Find the slope of the tangent line to the semi-circle $y = \sqrt{r^2 - x^2}$ at the point where $x = a$.

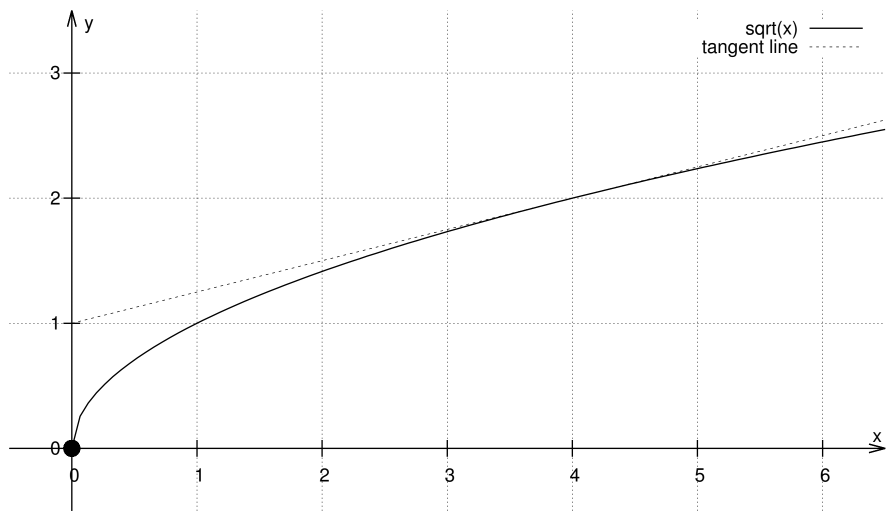

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Figure 1: Checking that we have found the tangent line