1 Lecture 14: The product and quotient rule

1.1 Outline

- The product rule
- The reciprocal rule
- The quotient rule.

1.2 The derivative of a product

Theorem 1 *Suppose that f and g are two functions which are differentiable at a point x, then fg is differentiable at x and*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Proof. The proof depends on rewriting the difference quotient for fg in terms of the difference quotients for f and g. This depends on the trick of adding and subtracting $f(x)g(x+h)$ as follows

$$\begin{aligned}
\frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h}.
\end{aligned}$$

We know that the difference quotients for f and g have a limit as h tends to zero. Since g is differentiable at x, it is continuous and we have

$$\lim_{h\to 0} g(x+h) = g(x).$$

Thus we may use the rules for sums and products of limits to obtain that

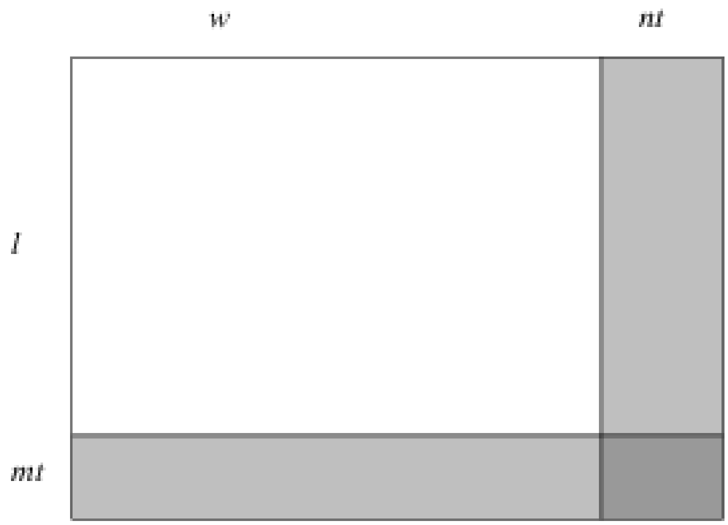
$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

∎

One way to understand this rule is to think of a rectangle whose length ℓ and width w are given by $\ell(t) = \ell + mt$ and $w(t) = w + nt$. Then the area will be given by

$$\ell(t)w(t) = (\ell + mt)(w + nt) = \ell w + (mw + \ell n)t + mnt^2.$$

At $t = 0$, the instantaneous rate of change of the area will be the coefficient of t, $mw + \ell n$. Since m is the rate of change of ℓ and n is the rate of change of w, the rate of change of the product is exactly what we see in the product rule. The picture shows how the increase in area of a rectangle is related to the sidelengths of the rectangle.



Example. Compute the derivative $f(x) = (1+2x)^2 e^x$.

Solution. At the moment, we do not know how to differentiate the function $(1+2x)^2$. However, if we expand the square, we can write

$$(1+2x)^2 e^x = (1+4x+4x^2)e^x.$$

We use the Leibniz notation,

$$\begin{aligned}
\frac{d}{dx}((1+4x+4x^2)e^x) &= (\frac{d}{dx}(1+4x+4x^2))e^x + (1+4x+4x^2)\frac{d}{dx}e^x \\
&= (4+8x)e^x + (1+4x+4x^2)e^x \\
&= (5+12x+4x^2)e^x.
\end{aligned}$$

∎

Example. Find the derivative of x^5 by writing $x^5 = x^4 \cdot x$ and applying the product rule.

Solution. We write $x^5 = x^4 \cdot x$ and apply the product rule,

$$\frac{d}{dx}x^5 = \frac{d}{dx}(x^4 \cdot x) = x^4 \frac{d}{dx}x + (\frac{d}{dx}x^4)x.$$

Computing the derivatives gives

$$4x^3 \cdot x + x^4 \cdot 1 = 5x^4.$$

∎

1.3 Reciprocals

We find the derivative of a reciprocal or the multiplicative inverse of a function.

Theorem 2 *If g is differentiable at x and $g(x) \neq 0$, then $1/g$ is differentiable at x and we have*

$$\left(\frac{1}{g}\right)'(x) = \frac{-g'(x)}{g(x)^2}.$$

Proof. We write out the difference quotient for $1/g$, obtain a common denominator and simplify to express it in terms of the difference quotient for g,

$$\begin{aligned} \frac{1}{h}\left(\frac{1}{g(x+h)} - \frac{1}{g(x)}\right) &= \frac{1}{h}\frac{g(x)}{g(x)g(x+h)} - \frac{g(x+h)}{g(x)g(x+h)} \\ &= \frac{-1}{h}(g(x+h) - g(x))\frac{1}{g(x+h)g(x)}. \end{aligned}$$

Now we may use the limit laws and that $1/g$ is continuous at x to write

$$\begin{aligned} \left(\frac{1}{g}\right)'(x) &= -\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \cdot \lim_{h\to 0}\frac{1}{g(x)g(x+h)} \\ &= \frac{-g'(x)}{g(x)^2}. \end{aligned}$$

∎

Example. Use this rule to find the derivative

$$\frac{d}{dx}\frac{1}{x^4}.$$

Solution. $\frac{d}{dx}\frac{1}{x^4} = \frac{-4x^3}{(x^4)^2} = \frac{-4}{x^5}$. ∎

We can use the reciprocal rule to extend the power rule to negative exponents.

Example. Use the reciprocal rule to find the derivative

$$\frac{d}{dx}x^{-n}, \qquad \text{for } n = 1, 2, 3, \ldots.$$

Solution. $\frac{d}{dx}\frac{1}{x^n} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}$. ∎

1.4 Quotient rule

Finally we give the quotient rule. Note that it is often simpler to rewrite a quotient as a product and avoid the quotient rule.

Theorem 3 *If f and g are differentiable at x and $g(x) \neq 0$, then f/g is differentiable at x and*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Proof. We may prove this writing $f/g = f \cdot \frac{1}{g}$ and using the product and the reciprocal rule.

$$\left(\frac{f}{g}\right)'(x) = (f\frac{1}{g})'(x) = f'(x)\frac{1}{g(x)} + f(x)\frac{-g'(x)}{g(x)^2}$$

We may simplify this last expression by obtaining a common denominator.

$$f'(x)\frac{1}{g(x)} + f(x)\frac{-g'(x)}{g(x)^2} = f'(x)\frac{g(x)}{g(x)^2} + f(x)\frac{-g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

∎

Example. Find the tangent line to

$$f(x) = \frac{x^2+3}{x^2-3}$$

at $x = 1$.

Solution. The tangent line will pass through the point $(2, f(2)) = (2, 7)$. We need the derivative of f to compute the slope. We use the quotient rule to find the derivative of f,

$$\begin{aligned} f'(x) &= \frac{(\frac{d}{dx}(x^2+3))(x^2-3) - (x^2+3)\frac{d}{dx}(x^2-3)}{(x^2-3)^2} \\ &= \frac{2x(x^2-3) - (x^2+3)2x}{(x^2-3)^2} \\ &= \frac{-12x}{(x^2-3)^2}. \end{aligned}$$

At 2, we have $f'(2) = -24$. Thus the tangent line has the equation

$$y - 7 = -24(x-2).$$

We simplify this to give $y = -24x + 55$ as the equation of the tangent line. ∎

Exercise. Find the derivative of $f(x) = \frac{1+2x}{1-2x}$.