1 Lecture 15: Rates of change

1.1 Outline

- Rates of change, velocity, acceleration.
- Marginal rates of change, interpreting the derivative as the response to a unit change
- Equation of motion for objects falling under constant gravity
- Interpreting position and velocity graphs

1.2 Rates of change

If $f(t)$ is a function depending on time, then we can write down the average rate of change on an interval $[t_1, t_2]$,

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

We have defined the instantaneous rate of change at a time t as the limit

$$f'(t) = \lim_{r \to t} \frac{f(r) - f(t)}{r - t}.$$

This is also called the derivative. Thus, instantaneous rate of change is another name for the derivative. In applications the name rate of change is more descriptive.

If $s(t)$ gives the position of a particle as it moves along a line, then

$$v(t) = \lim_{k \to 0} \frac{s(t+k) - s(t)}{k}$$

gives the instantaneous velocity. If position is measured in meters and time in seconds, the velocity v is measured in meters/second. We will abbreviate meters by m and seconds by s so that the units for velocity are m/s.

As a second example, let $v(t)$ be a function which gives the velocity of a particle, measured in meters per second, at time t seconds. Then the instantaneous rate of change of velocity is

$$a(t) = \lim_{r \to t} \frac{v(r) - v(t)}{r - t}.$$

This is called *acceleration* and if velocity is measured in meters per second (or m/s) and time is measured in seconds, then the units for acceleration will be

$$\frac{\mathrm{m/s}}{\mathrm{s}} = \mathrm{m/s}^2.$$

If $a > 0$, then the velocity is increasing and if $a < 0$, then velocity is decreasing. We will also consider speed which is the absolute value of velocity.

1.3 The derivative as a marginal rate

If we approximate the derivative $f'(a)$ by a difference quotient $\frac{f(a+1)-f(a)}{1}$ we have

$$f(a+1) \approx f(a) + f'(a).$$

At the moment, we have absolutely no way to discuss the error committed in this approximation. Nonetheless, this is an approximation that is quite common in application areas. From this approximation, we see that the derivative corresponds to the increase resulting from a one unit increase. This is often called a marginal rate.

Example. If $f(x) = \sqrt{x}$, use the approximation to estimate $f(82)$. Can you give simple approximations for $f(80)$ and $f(83)$. Note that $f(81)$ is very easy to compute.

Solution. We have $f(82)$ is approximately $f(81) + f'(81)$. Computing $f'(x) = 1/(2\sqrt{x})$ and thus $f(81) + f'(81) = \sqrt{81} + 1/(2\sqrt{81}) = 9 + 1/18 = 163/18$. Checking that $(163/18)^2 \approx 82.003$ this seems to a very good approximation.

A moment's thought and recalling that $f'(x)$ is a rate of change suggests that

$$f(80) \approx f(81) - f'(81) = 161/18 \qquad f(83) \approx f(81) + 2f'(81) = 164/18 = 82/9.$$

Checking we see that $(161/18)^2 \approx 80.003$ and $(164/18)^2 \approx 83.003$. Thus these approximations are quite good. ∎

1.4 Object falling under gravity

We now give a simple, but very important equation for the height of an object moving under the influence of gravity. At the moment, we can only write down the equation. By the end of the semester, we will know how to recover the height of an object h from the acceleration a.

Example. If an object is thrown near the surface of the Earth, its height at time t units after being thrown is given by

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + h_0.$$

Show that the acceleration is the constant $-g$ and find the velocity and height at time $t = 0$.

Solution. Differentiating twice, we have

$$v(t) = v_0 - gt \qquad \text{and} \qquad a(t) = -g.$$

Thus the acceleration is constant. If we set $t = 0$ we find $v(0) = v_0$ and $h(0) = h_0$. ∎

If we measure height in meters and time in seconds, then $g = 9.8$ m/s^2. Note that the negative sign in our expression for the height indicates that gravity is pulling the particle down or in the negative direction.

Example. Suppose an object is thrown from the ground with an initial velocity of 30 m/s^2. When will it return to the ground? What is its velocity when it hits the ground?

Solution. We have the height at time t is given by $h(t) = -4.9t^2 + 30t$ meters. If we solve $h(t) = 0$, then $0 = t(30 - 4.9t)$ so the object has $h = 0$ when $t = 0$ and $t = 30/4.9 \approx 6.1s$. At this time $v(30/4.9) = -9.8(30/4.9)+30 = -60+30 = -30$m/s. Thus the velocity is -30m/s. ∎

Example. Suppose that we throw a ball up in the air and it returns to ground level after 4 seconds. a) What is the initial velocity? b) What is the greatest height?

Solution. We may assume that the initial height is zero, $h_0 = 0$, and we know that $h(4) = -\frac{1}{2}g4^2 + v_0 4 = 0$. Solving this equation for v_0 gives

$$v_0 = \frac{1}{2}g \cdot 4.$$

When $g = 9.8$ m/s^2, this gives the numerical value $v_0 = \frac{1}{2} \cdot 9.8\text{m/s}^2 \cdot 4\text{s} = 19.6$ m/s.

We know the maximum height will occur when the velocity is zero. Solving $v(t) = v_0 - gt = 0$ gives the maximum occurs when $t = t_{max} = v_0/g$. The maximum height will be $h(t_{max}) = -\frac{1}{2}g(v_0/g)^2 + v_0 v_0/g = -\frac{1}{2}v_0^2/g$. Since $v_0 = 19.6\,\text{m/s}$ and $-g = 9.8\,\text{m/s}^2$ the maximum height is 19.6 m. ∎

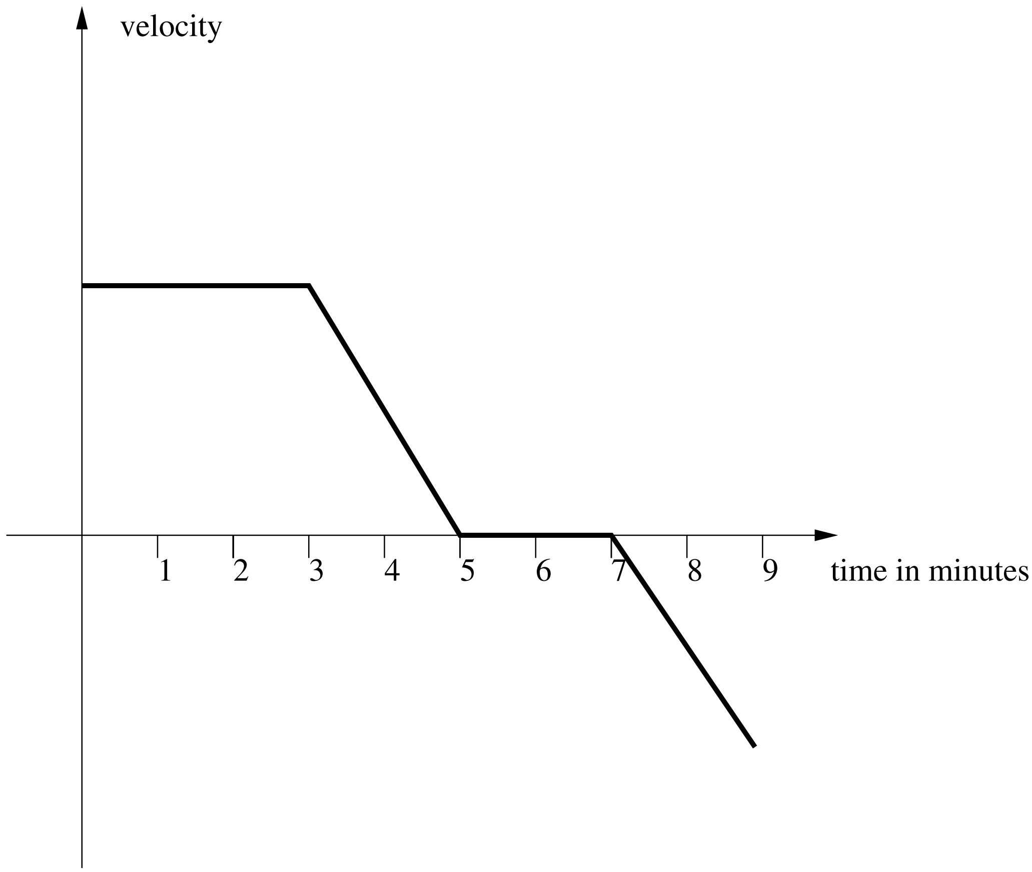
Example. Suppose that an object moves along the real line so that its position at time t is $s(t) = t^3 - 6t^2$. When is the object's velocity increasing? When is the speed increasing?

Solution. We have $s(t) = t^3 - 4t^2$ and thus $s'(t) = v(t) = 3t^2 - 12t$ and $a(t) = v'(t) = 6t - 12$. We have $a > 0$ for $t > 2$ and $a < 0$ for $t < 2$.

The speed will be increasing if a) the velocity is positive and the acceleration is positive and b) the velocity is negative and the speed is negative. Since $v < 0$ for $0 < t < 4$, we have that the speed is increasing for $0 < t < 2$. The velocity is positive for $t > 4$ and the acceleration is positive there, also. So the speed is increasing for $t > 4$. ∎

1.5 Interpreting a velocity graph

Example. The graph below gives the velocity of Radar as he walks along the number line. Use the graph below to answer the following questions: a) When is the Radar resting? b) When is Radar moving to the right? c) When is Radar's velocity increasing? d) What information do you need to sketch a graph of Radar's position?



Solution. a) The velocity is zero between 5 and 7 minutes. b) The velocity is positive from 0 to 5 minutes. c) Never. d) A starting point. We might also need units for the vertical axis. ∎

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