1 Lecture 19: Inverse functions, the derivative of $\ln(x)$.

1.1 Outline

- The derivative of an inverse function
- The derivative of $\ln(x)$.
- Derivatives of inverse trigonometric functions

1.2 The graph of inverse function

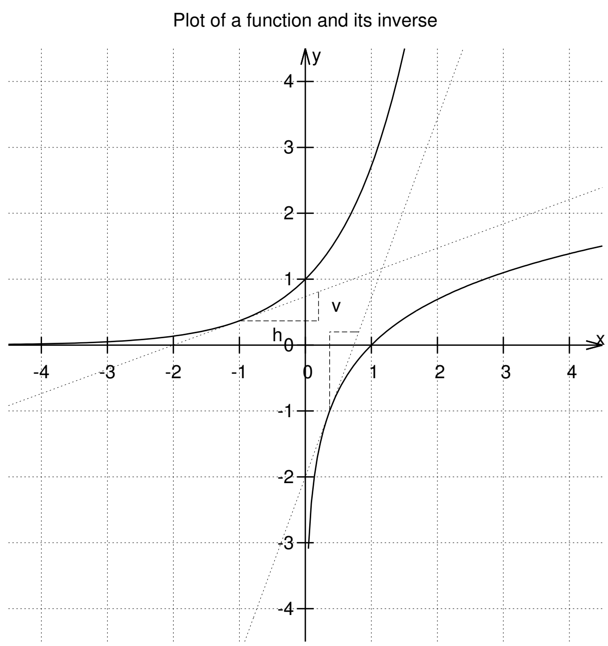

Figure 1: The tangent line to a function and its inverse

We consider the graph of a function f and let $(a, f(a)) = (a, b)$ be a point on the graph. We recall that if we graph a function f which is one-to-one, we may find the graph of the inverse function by interchanging the x and y coordinates. Another name for interchanging the x and y coordinates is to reflect in the line $y = x$. Thus $(b, a) = (b, f^{-1}(b))$ will be a point on the graph of f^{-1}. We may find the tangent line to the graph at f^{-1} at $(b, f^{-1}(b))$, by reflecting the tangent line at $(a, f(a)) = (f^{-1}(b), b)$. This interchanges the rise and the run and the reflected line has slope is $1/m$. See Figure 1. Thus, we may find the derivative of the inverse function in terms of the derivative of f as

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}. \tag{1}$$

This will hold provide f is one-to-one, differentiable at $f^{-1}(b)$ and $f'(f^{-1}(b))$ is not zero. The formula for the derivative of an inverse function (1) may seem rather complicated, but it helps to remember that the tangent line to the graph of f^{-1} at a point $(b, f^{-1}(b))$ corresponds to the tangent line of the graph of f at $(f^{-1}(b), b)$. We will see that the formula is easy to use to find find derivatives of the logarithm and inverse trig functions.

Example. The function f is one-to-one and differentiable. The tangent line to the graph of $y = f(x)$ at $x = 2$ is $y = 3x + 2$. Find the tangent line to the graph of f^{-1} at $x = 8$.

Solution. Since $y = 3x+2$ is the tangent line to the graph of f, it passes through the point $(2, f(2))$ and has slope $f'(2)$. Thus, we have $f(2) = 3 \cdot 2 + 2 = 8$ and $f'(2) = 3$. Since $f(2) = 8$, we know $f^{-1}(8) = 2$ and from (1), we have that

$$(f^{-1})'(8) = \frac{1}{f'(2)} = \frac{1}{3}.$$

Thus the tangent line to the graph of f^{-1} at 8 is $y - 2 = \frac{1}{3}(x - 8)$ which simplifies to

$$y = \frac{1}{3}x - \frac{2}{3}.$$

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1.3 Some useful derivatives

We recall that the natural logarithm, $\ln(x)$ is defined as the inverse of the exponential function e^x. If we use (1) with $f(x) = e^x$ and $f^{-1}(x) = \ln(x)$, we obtain

$$\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

Where we have used that $e^{\ln(x)} = x$ and $\frac{d}{dx}e^x = e^x$.

Example. Find the derivative of

$$f(x) = x\ln(x) - x.$$

The function $\sqrt[n]{x} = x^{1/n}$ is the inverse of the function $f(x) = x^n$ where if n is even we must restrict the domain of f to be the set $\{x : x \geq 0\}$. If n is odd, then f is one-to-one on the whole real line.

Example. Use the rule for the derivative of the inverse function to find the derivative of $g(x) = x^{1/n}$.

Solution. We let $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Using (1), we have

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

At least when $x \neq 0$ so that $f'(x) \neq 0$. Since the domain of g is $x \geq 0$ when n is even, we can only expect to find the derivative for $\sqrt[n]{x}$ when $x > 0$ for n even. To summarize, we have

$$\frac{d}{dx}x^{1/n} = \begin{cases} \frac{1}{n}x^{\frac{1}{n}-1}, & x \neq 0 \text{ and } n \text{ odd} \\ \frac{1}{n}x^{\frac{1}{n}-1}, & x > 0 \text{ and } n \text{ even} \end{cases}$$

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This gives the power rule for all exponents of the form $1/n$.

Exercise. Use the previous result and the chain rule to verify the power rule for rational exponents. That is find

$$\frac{d}{dx}x^{m/n} = \frac{d}{dx}(x^{1/n})^m.$$

We have now proven the power rule for all rational exponents. The rule is true and we may use the rule for any real exponent. However, the proof for the general case is very different. The next problem shows how to find the derivative of x^r using the properties of the logarithm and the exponential function. The exercise also gives some practice with the chain rule.

Exercise. To prove the power rule for real exponents, we may argue as follows. If $y = x^r$, then we have $\ln(y) = \ln(x^r) = r\ln(x)$. Thus we have $y = e^{\ln(y)} = e^{r\ln(x)}$. This provides a careful definition of x^r for any real number. Use the chain rule to find

$$\frac{d}{dx}e^{r\ln(x)}, \qquad x > 0.$$

1.4 Inverse trigonometric functions

Finally, we find the derivatives of the inverse trigonometric functions. For these functions, we will need to use trigonometric identities to simplify the result of (1).

We begin by finding the derivative of $\sin^{-1}(x)$. This function is often written as arcsin, but we will not use this notation in this course. Please remember that $\sin^{-1}$ is in the inverse function. The reciprocal or multiplicative inverse is $1/\sin(x) = \csc(x)$.

If $f(x) = \sin(x)$ and $f^{-1}(x) = \sin^{-1}(x)$, then using our formula for the inverse function (1) we have

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))}.$$

We have a problem that we need to simplify $\cos(\sin^{-1}(x))$. We recall that the range of $\sin^{-1}$ is the interval $[-\pi/2, \pi/2]$ and that $\cos(\theta) \geq 0$ for $\theta \in [-\pi/2, \pi/2]$. If we solve the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ for $\cos(\theta)$ and use that $\cos(\theta) \geq 0$, we obtain that

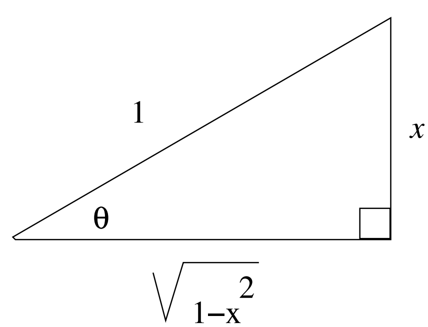
$$\cos(\sin^{-1}(x) = \sqrt{1 - \sin^2(\sin^{-1}(x))} = \sqrt{1 - x^2}.$$

Using this we obtain that

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1. \tag{2}$$

We cannot use (1) when $x = \pm 1$ since the denominator is zero and in fact $\sin^{-1}$ is not differentiable for $x = \pm 1$.

A second approach to the simplification is to draw a right triangle with angle $\theta = \sin^{-1}(x)$, opposite side x and hypotenuse 1. Then Pythagoras's theorem gives us that $\cos(\theta) = \sqrt{1-x^2}$.



Example. Use (1) to find the derivative of $\tan^{-1}(x)$.

Solution. We will need the identity

$$1 + \tan^2(x) = \sec^2(x). \tag{3}$$

This may be established by dividing both sides of the identity $\sin^2(x) + \cos^2(x) = 1$ by $\cos^2(x)$.

Since the derivative of $\tan(x)$ is $\sec(x)$, we have from (1) that

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{\sec^2(\tan^{-1}(x))}.$$

We use the identity (3) to simplify the denominator $\sec^2(\tan^{-1}(x)) = 1+\sec^2(\tan^{-1}(x)) = 1 + x^2$ and find that

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}.$$

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Exercise. Find the derivatives of the remaining inverse trigonometric functions,

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}.$$

$$\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2} \qquad \frac{d}{dx}\csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

Of these, we will only need to remember the derivative of $\sec^{-1}(x)$.

Example. Find the derivative of $\sin^{-1}(1/x)$ and show that

$$\frac{d}{dx}\sin^{-1}(1/x) = \frac{d}{dx}\csc^{-1}(x).$$

Why is this?

Solution. Since $\csc(\theta) = 1/\sin(\theta)$, if $x = \csc(\theta)$, then $1/x = \sin(\theta)$ which implies that $\sin^{-1}(1/x) = \csc^{-1}(x)$. Thus the derivatives are equal. To compute the derivative, we use the chain rule

$$\frac{d}{dx}\sin^{-1}(1/x) = \frac{-1}{x^2}\frac{1}{\sqrt{1-(1/x)^2}} = \frac{-1}{x^2}\frac{1}{\sqrt{1/x^2}\sqrt{x^2-1}} = \frac{-1}{|x|\sqrt{x^2-1}}.$$

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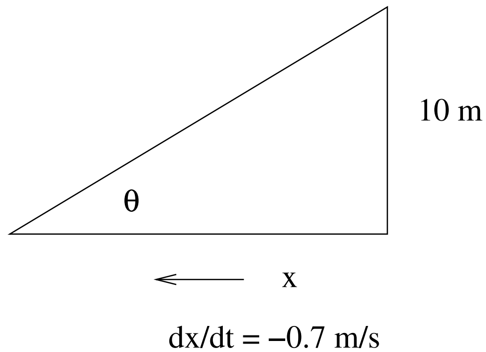
Example. Suppose that one leg of a right triangle is fixed at 10 meters and the other leg is decreasing at a rate of 0.7 meters/second. Find the rate of change of the angle opposite the fixed leg when the length of the varying leg is 20 meters.

Solution. We let $x(t)$ be a function of time which gives length of the varying leg of the triangle and sketch the triangle.



We may write $\theta = \tan^{-1}(10/x(t))$ and use the chain rule to find

$$\frac{d\theta}{dt} = \frac{d}{dt}\tan^{-1}(10/x(t)) = \frac{-10x'(t)}{x(t)^2}\frac{1}{1+(10/x(t))^2} = \frac{-10x'(t)}{x(t)^2+10^2}.$$

Simplifying and substituting the given values gives

$$\frac{d\theta}{dt} = \frac{7}{20^2+10^2} = 7/500 \text{ radians/second.}$$

Can you explain the units for the answer?

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