1 Lecture 21: Related rates

1.1 Outline

- Outline of problem-solving.
- Similar triangles, Pythagoras theorem,
- Examples

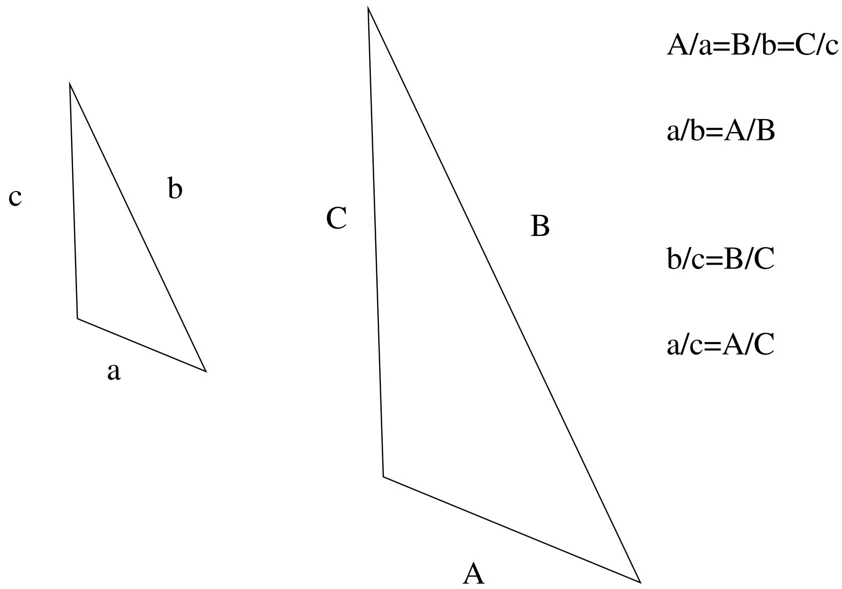
1.2 Steps to solve a problem

1. Draw a sketch, label everything with variables.
2. List given information about the variables and their rates of the change. Do not assign numerical values to variables that are changing.
3. Identify the information that is needed.
4. Write an equation relating the unknown variable to known quantities.
5. Differentiate.
6. Substitute values.
7. Answer the question.

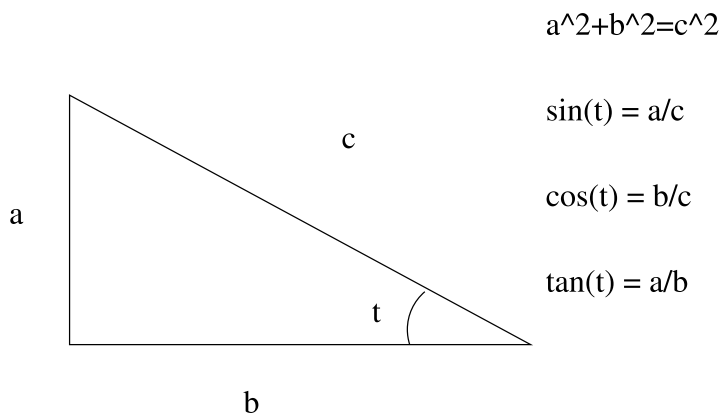
1.3 Geometric preliminaries

Many problems will involve triangles and we can define relationships among the sides of the variables.

The most common relations are ratios of corresponding sides in similar triangles

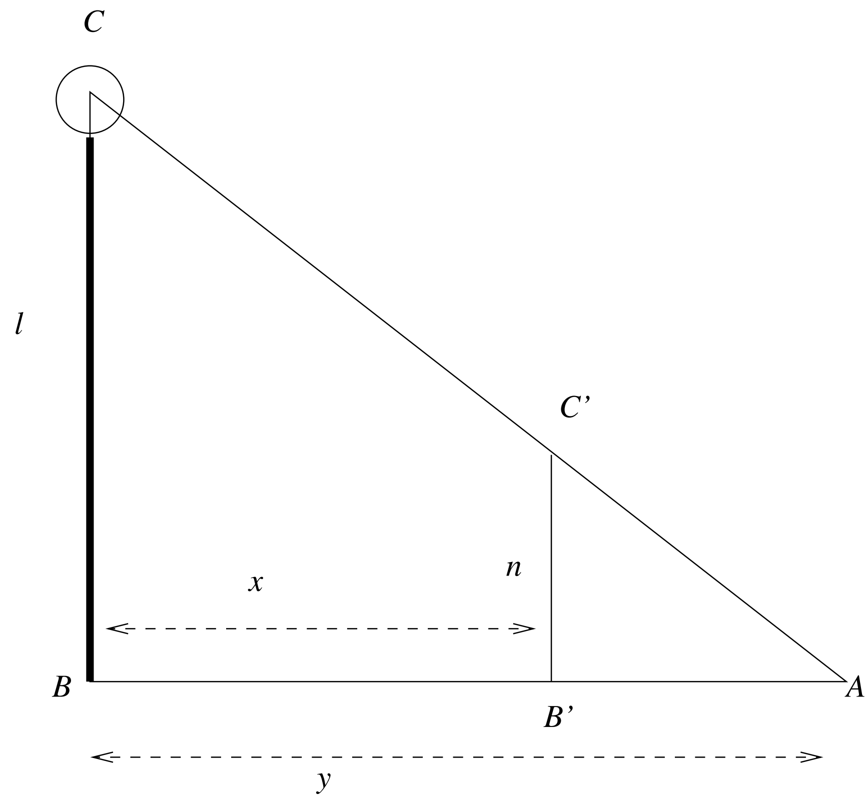


and the trigonometric functions which relate an angle and sides of a right triangle and the Pythagorean theorem



1.4 Examples

Example. A man who is two meters tall is walking away from a streetlight at a rate of 0.9 meters/second. The streetlight is 20 meters tall. How fast is the tip of his shadow moving when he is 15 meters from the light.



Solution. We draw two right triangles so that $l = 20$ m is the height of the light and $n = 2$ m is the height of the man. These lengths are fixed, so $dn/dt = dl/dt = 0$. The function $x = x(t)$ gives the distance of the man from the light and we are told that $dx/dt = 0.9$ m/s. We *do not* want to substitute $x = 15$ yet as this would not allow us to use the information that x is changing.

The problem asks to find the speed that the tip of the shadow is moving. As $y = y(t)$ is the distance from the base of the light to the tip of the shadow, we want to find dy/dt at the instant that $x = 15$.

To obtain a relation between the variables x, y, n and l, we use that the triangles ABC and $AB'C'$ are similar. (The primes do not indicate derivatives.) To see that these triangles are similar. We show that two pairs of angles are of the same measure and then the third pair of angles will be equal since the three angles sum to π radians. The angles at B and B' are right angles and the angle at A is common to both triangles and this gives of two pairs of equal angles. The ratio of lengths of corresponding sides AB' a and $B'C'$ and AB and BC gives us the equation,

$$\frac{y}{l} = \frac{y - x}{n}.$$

If we differentiate with respect to time and use that n and l are constants, we obtain

$$\frac{y'}{l} = \frac{y' - x'}{n}.$$

where x' and y' are derivatives with respect to time. Solving this equation for y' gives

$$y' = \frac{l}{l-n} x'.$$

Finally, we can substitute the given values to obtain

$$y' = \frac{20\text{m}}{(20-2)\text{m}} 0.9\text{m/s} = 1\text{m/s}.$$

Note that the answer is independent of the distance of the man from the streetlight. The distance 15 that appears in the problem is not needed to solve the problem.

The derivative y' represents the velocity of the tip of the shadow. Thus the shadow is moving at 1 m/s.

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Example. Suppose that we are inflating a spherical balloon so that its volume is increasing at a rate of 12 cubic centimeters/second. Find the rate of change of the radius when the radius is 10 cm.

Solution. We let r denote the radius in centimeters and V the volume of the balloon in cubic centimeters. We have that $V = \frac{4}{3}\pi r^3$. We are given that $dV/dt = 12\text{cm}^3/\text{s}$ and would like to find dr/dt when $r = 10$ cm.

We differentiate V with respect to t and obtain

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Solving this equation for dr/dt and substituting values gives

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}.$$

At the time when $r = 10$ cm, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi 10^2} 12 = \frac{3}{100\pi}\text{cm/s}.$$

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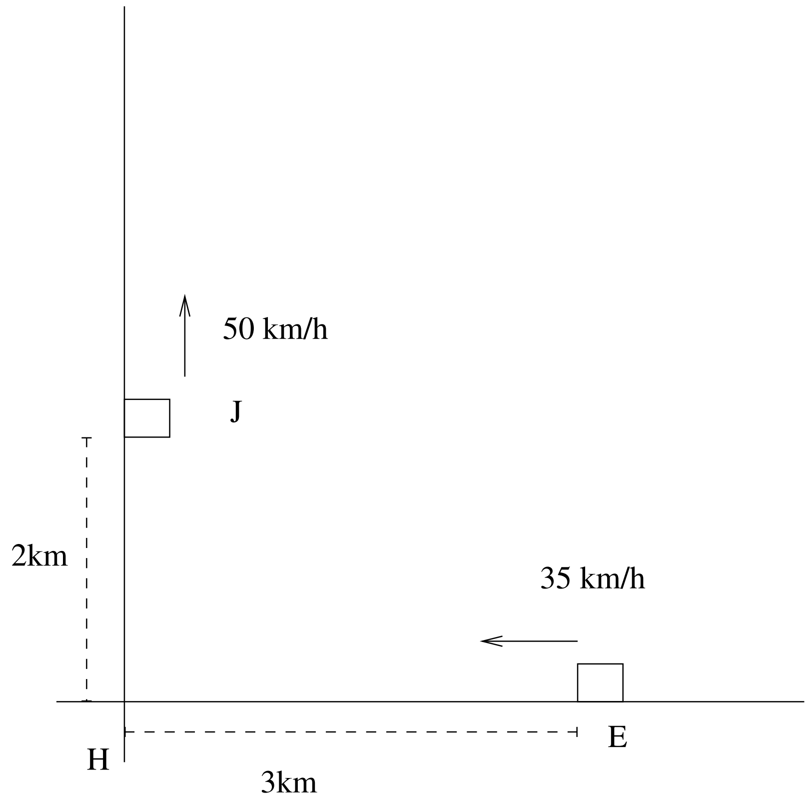
Example. Two roads meet at Haverhill Corner in a right angle. Joe is north of Haverhill Corner and is driving north away from the corner, while Elizabeth is east of Haverhill Corner and is driving west.

At a certain time, Joe is 2 kilometers from Haverhill corner and driving at a speed of 50 kilometers per hour, while Elizabeth is 3 kilometers from the corner and driving at a speed of 35 kilometers per hour. Is the distance between Joe and Elizabeth decreasing or increasing?

Solution. We make a sketch of the corner and the roads as follows. We let $x(t)$ be the distance from the corner to Elizabeth and $y(t)$ be the distance from the corner to Joe.



At the time of interest, we have

$$x = 3, \qquad x' = -35, \qquad y = 2, \qquad y' = 50. \tag{1}$$

The distance between Elizabeth and Joe is given by

$$d = \sqrt{x^2 + y^2}.$$

But it is simpler to work with the square of the distance, $d^2 = x^2 + y^2$. If we assume d, x, and y are all functions of t, and let $'$ be the derivative with respect to t, we have

$$2dd' = 2xx' + 2yy'.$$

Substituting the values given in (1), we have

$$d' = \frac{1}{\sqrt{13}}(3 \cdot -35 + 2 \cdot 50) = -5/\sqrt{13}\text{km/h}.$$

Thus, the distance between the two is decreasing at the rate of $-5/\sqrt{13}$km/h.

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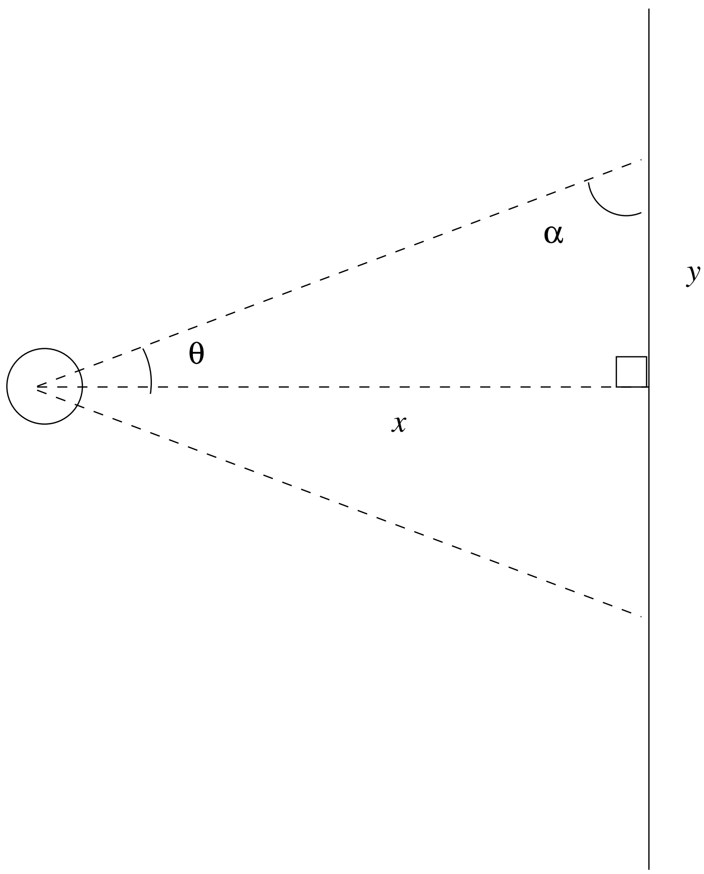
For our last example, we will use a trigonometric function to give the relation between an angle and the sides of a right triangle.

Example. Suppose that a light is 20 meters from a wall and rotates at 2 radians/minute. Find the speed of the light along the wall when the light beam forms an angle of $\pi/3$ with the wall.



Solution. As in the sketch above, we let $a = 20$ m denote the distance from the wall to the ball and this distance is fixed, $a' = 0$. We let $\theta = \theta(t)$ we the angle between line perpendicular to the wall that passes through the light and the light beam and α the angle between the wall and light beam. We are given that $\theta' = 2$ radians/minute (or just 2 minute^{-1}). We want to find the dy/dt at the instant when $\alpha = \pi/3$. Since the angles θ and α sum to $\pi/2$, we have that $\theta = \pi/6$ when $\alpha = \pi/3$.

We may begin with the relation $\tan(\theta) = y/a$ and differentiate both sides with respect to t to obtain

$$\theta' \sec^2(\theta) = y'/a.$$

Solving for y' and substituting values

$$y' = a\theta'/\cos^2(\pi/6) = 20 \cdot 2/(\sqrt{3}/2)^2.$$

We obtain $y' = \frac{160}{3}$ m/minute. Note that angles are ratio of lengths, so the proper units for θ' is minute$^{-}1$.

Observe that the problem does not make clear which direction the light is rotating. There is another position when the light beam will make an angle of $\pi/3$ with the wall. Will changing the direction of the light beam or looking at the other point change the speed? ∎

March 6, 2014