1 Lecture 26: Mean value theorem and monotonicity

1.1 Outline

- The mean value theorem
- Test for monotonicity
- First derivative test for local extreme

1.2 The mean value theorem

Definition. We say that a function f is increasing on an interval I if whenever x and y are in I, then $f(x) < f(y)$.

Exercise. Write out a definition for f to be decreasing on an interval.

If we have a function where the derivative is positive, we expect that the function is increasing. The derivative at a point only depends on values of the function in arbitrarily small intervals near a point, while to say that a function is increasing requires us to look at points that are far apart. Using the derivative to obtain information about the function is surprisingly difficult. The mean value theorem is the tool we use to make the connection between the derivative and the function.

Theorem 1 (Mean value theorem) *Suppose that f is a function which is continuous on a closed interval $[a, b]$ and differentiable on an open interval (a, b), then there exists a number c in (a, b) so that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

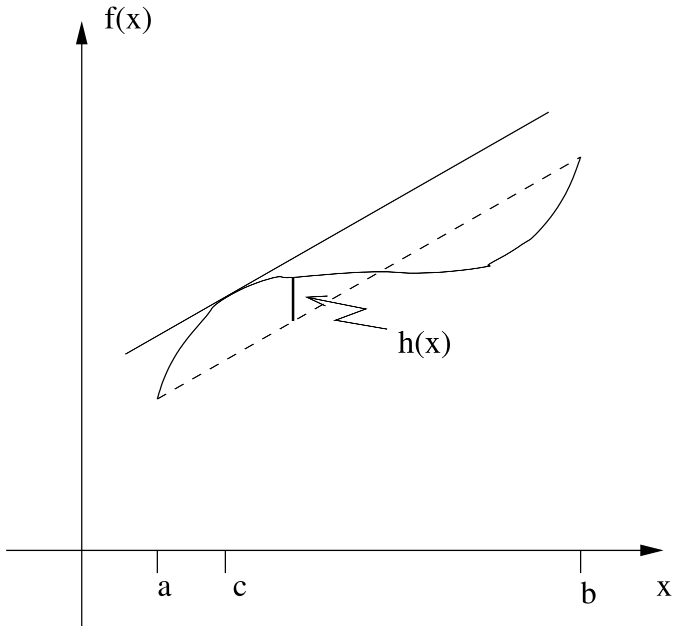
This theorem tells us that if we join two points $(a, f(a))$ and $(b, f(b))$ by a line segment, there will be at least one point where the derivative equals the slope of this line segment.

Proof. We joint let $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$. It is easy to see that $h(a) = h(b) = 0$ and that h satisfies the hypotheses of Rolle's theorem. Thus Rolle's theorem tells us that $h'(c) = 0$ for some c in (a, b).

But $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ so if $h'(c) = 0$, then

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The proof is summarized in the picture



∎

Example. Suppose that f is a differentiable function on the real line, $f(1) = 2$ and $-3 \le f(x) \le 4$, what are the largest possible values for $f(5)$ and $f(-5)$?

Solution. Since f is differentiable for all real numbers, f is continuous and we may use the mean-value theorem.

To find an upper bound for $f(5)$, we use the mean value theorem to conclude that the slope of the line joining $(1, f(1))$ and $(5, f(5))$ is equal to the value of the derivative at some point, $f'(c)$. As $-3 \le f'(c) \le 5$, we conclud

$$-3 \le \frac{f(5) - f(1)}{5 - 1} \le 4.$$

Simplifying this inequality, we find

$$-12 + f(1) \le f(5) \le 8 + f(1).$$

Thus, we have $f(5) \le 8 + f(1) = 10$.

To estimate $f(-5)$, we apply the mean value theorem on the interval $[-5, 1]$ and conclude that

$$-3 \le \frac{f(-5) - f(1)}{-5 - 1} \le 4.$$

Simplifying the inequality gives

$$18 + f(1) \ge f(-5) \ge -24 + f(1).$$

Remember to switch the direction of the inequality signs when we multiply by a negative number! Thus we conclude that

$$f(-5) \leq 18 + 2 = 20.$$

In each case, we see that f can obtain the value if f is a linear function. ∎

Exercise. With the same information as in the previous example, find the smallest possible values for $f(-5)$ and $f(5)$.

1.3 First derivative test for increasing and decreasing behavior

As promised the mean value theorem gives us the relation between the derivative and the graph of a function.

Theorem 2 *Let f be a differentiable function on an open interval (a, b).*

- *If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).*
- *If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).*

Proof. If $a < x < y < b$ and $f'(t) > 0$ for all t in (a, b), then the mean value theorem tells us that

$$\frac{f(y) - f(x)}{y - x} > 0.$$

Simplifying the inequality gives that $f(x) < f(y)$.

We leave the case when f' is negative for the reader to work out. ∎

If we know the sign of the first derivative, then it is easy to test critical points to determine if they are local extreme values.

Theorem 3 *Assume that f is a differentiable function and c is a critical point for f. If there is an open interval (a, b) which contains c and*

1. *if $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$, then f has a local minimum at c.*
2. *if $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$, then f has a local maximum at c.*

This result is obvious. If f is decreasing to the right of c and increasing to the left of c, then f must have a local maximum at c.

Exercise. Let $f(x) = x^3 + 3x^2 - 9x$. Find the intervals where f is increasing and decreasing and use this information to determine if f has a local maximum or local minimum at each critical point.

Solution. We differentiate f and factor to find that

$$f'(x) = 3x^2 + 6x - 9 = 3(x-1)(x+3).$$

The critical points will be at $x = 1$ and $x = -3$. The graph of f' is a parabola opening up thus

- $f'(x) > 0$ and thus f is increasing on $(-\infty, -3)$
- $f'(x) < 0$ and thus f is decreasing on $(-3, 1)$
- $f'(x) > 0$ and thus f is increasing on $(1, \infty)$

Since f is increasing to the left of -3 and decreasing to the right of -3, f has a local maximum at -3.

Since f is decreasing to the left of 1 and increasing to the right of 1, f has a local minimum at 1. ∎

Exercise. Let $f(x) = e^{-x^2+2x}$. Find the intervals where f is increasing and decreasing and use this information to determine if f has a local maximum or local minimum at each critical point.

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