1 Lecture 28: Limits at infinity

1.1 Outline

- Informal definition
- Limits of powers
- Limits of rational function
- Further limits

1.2 Definition

In Chapter 1, we considered the limit of a function at a finite value a, $\lim_{x\to a} f(x)$. Today we consider a similar notion except that we consider the behavior of $f(x)$ for values of x which are arbitrarily large and positive, or large and negative.

Definition. Let f be a function defined on a domain that contains (a, ∞) for some real number a. We say that the *limit of f as x approaches infinity is L* or

$$\lim_{x\to+\infty} f(x) = L$$

if we can make $f(x)$ arbitrarily close to L by making x arbitrarily large and positive.

Let f be a function defined on a domain that contains $(-\infty, a)$ for some real number a. We say that the *limit of f as x approaches negative infinity is L* or

$$\lim_{x\to-\infty} f(x) = L$$

if we can make $f(x)$ arbitrarily close to L by making f arbitrarily large.

Example. Find

$$\lim_{x\to\infty} \frac{1}{x}, \qquad \lim_{x\to-\infty} \frac{1}{x^2}, \qquad \lim_{x\to-\infty} \frac{1}{\sqrt{x}}.$$

Solution. We know that the reciprocal of a very large number is very small. A graph helps to confirm that

$$\lim_{x\to\infty} \frac{1}{x} = 0, \qquad \lim_{x\to-\infty} \frac{1}{x^2} = 0.$$

For the third limit $\sqrt{x}$ is not defined for negative values of x. We cannot give the behaviour as x approaches ∞ as $1/\sqrt{x}$ is not defined there. ∎

The limits at infinity give the behavior of the function for large values. Thus the values of $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are *horizontal asymptotes* for the function f.

We can also consider the four different possibilities $\lim_{x\to\pm\infty} f(x) = \pm\infty$. For example, we say that

$$\lim_{x\to\infty} f(x) = \infty$$

if we may make the values $f(x)$ positive and as large as we like by choosing x large and positive.

Exercise. Write out definitions of the other three cases.

Example. Find

$$\lim_{x\to\infty} x^3, \qquad \lim_{x\to-\infty} x^2, \qquad \lim_{x\to-\infty} x^5.$$

Solution. If x large and positive, then x^3 is also large and positive. Thus $\lim_{x\to\infty} x^3 = \infty$.

If x large and negative, then x^2 is large and positive, since an even power is always positive. Thus $\lim_{x\to-\infty} x^2 = \infty$.

If x large and negative, then x^5 is large and negative since an even power of a negative number is negative. Thus $\lim_{x\to-\infty} x^5 = -\infty$. ∎

If the limits involved are finite, we may use the limit laws for sums, products and quotients as before. If one of the limits is infinity, we must be careful. In particular, we cannot use limit laws for expressions of the form $0 \cdot \infty$ or ∞/∞.

Example. Let c be your favorite number. Find functions f and g so that

$$\lim x \to \infty f(x) = 0, \qquad \lim_{x\to\infty} g(x) = \infty, \qquad \lim_{x\to\infty} f(x)g(x) = c.$$

Solution. A simple way to make examples is to put $f(x) = c/x$, $g(x) = x$. Then it is easy to check that the functions f and g have the desired properties. ∎

2 Limits of rational functions at infinity

Recall that a rational function R is a function that is a quotient of polynomials, $R(x) = P(x)/Q(x)$ where P and Q are polynomials.

The next theorem describes how to take limits at infinity for rational functions.

Theorem 1 *Suppose that $P(x) = p_n x^n + \ldots + p_0$ and $Q(x) = q_m x^m + \ldots + q_0$, then*

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \frac{p_n}{q_m} \lim_{n\to\infty} x^{n-m}.$$

A similar result holds for limits at $-\infty$.

Proof. We consider $R(x)$ and factor out the term with the highest power in the numerator and the denominator to write

$$R(x) = x^{n-m} \frac{p_n + p_{n-1}x^{-1} + \ldots p_0 x^{-n}}{q_m + q_{m-1}x^{-1} + \ldots q_0 x^{-m}}.$$

By the limit laws, we have

$$\lim_{x\to\infty} \frac{p_n + p_{n-1}x^{-1} + \ldots p_0 x^{-n}}{q_m + q_{m-1}x^{-1} + \ldots q_0 x^{-m}} = p_n/q_m$$

where p_n and q_m are assumed to be nonzero. Thus we can write

$$\lim_{x\to\infty} R(x) = \lim_{x\to\infty} x^{n-m} \lim_{x\to\infty} \frac{p_n + p_{n-1}x^{-1} + \ldots p_0 x^{-n}}{q_m + q_{m-1}x^{-1} + \ldots q_0 x^{-m}} = \frac{p_n}{q_m} \lim_{x\to\infty} x^{n-m}.$$

∎

Example. Find the limits

$$\lim_{x\to-\infty} \frac{x^3+1}{2x^2-x}, \qquad \lim_{x\to\infty} \frac{x^2+1}{2x^3-x}.$$

Solution.

$$\lim_{x\to-\infty} \frac{x^3+1}{2x^2-x} = \lim_{x\to-\infty} x \lim_{x\to-\infty} \frac{1+x^{-3}}{2-x^{-1}} = -\infty.$$

$$\lim_{x\to\infty} \frac{x^2+1}{2x^3-x} = 0.$$

∎

Finally, we give an example where some simplification is needed before we take the limit.

Example. Find

$$\lim_{x\to\infty} x - \frac{x^2-2x}{x+1}.$$

Solution. Simplifying we have that

$$x - \frac{x^2 - 2x}{x+1} = \frac{x^2 + x - (x^2 - 2x)}{x+1} = \frac{3x}{x+1}.$$

Now this is a standard limit of a rational function and it is easy to see.

$$\lim_{x \to \infty} \frac{3x}{x+1} = 3.$$

∎

3 Further examples

We close with a variety of limits.

Example. Find

$$\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}}, \qquad \lim_{t \to \infty} \arctan(\frac{1+t}{1-t}).$$

Solution. For the first limit, we argue as with rational functions, but we need to be remember that $\sqrt{x^2}$ is always positive and thus $\sqrt{x^2} = |x|$,

$$\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to -\infty} \frac{x}{|x|} = \lim_{x \to -\infty} \frac{1}{\sqrt{1 + 1/x^2}} = -1.$$

For the second limit, we first note that $\lim_{t \to \infty} \frac{1+t}{1-t} = -1$. Then since the arctan is continuous at -1, we have

$$\lim_{t \to -\infty} \arctan(\frac{1+t}{1-t}) = \arctan(-1) = -\pi/4.$$

∎

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