1 Lecture 29: l'Hôpital's rule

1.1 Outline

- The problem and what we know.
- Statement of L'Hopital's rule
- Examples
- Limits at infinity, growth of the exponential function
- The indeterminate form $0 \cdot \infty$.
- e

1.2 What we know.

We have discussed limits before and used the idea of a limit to continuity for functions and the derivative. We can use this information to evaluate a number of limits.

The limits

$$\lim_{x \to \pi} \cos(x)/(1+x^2) = -1/2 \text{ and } \lim_{x \to 0} \frac{e^x}{1+x} = 1$$

are limits that can evaluated by the direct substitution rule.

We also can evaluate limits by recognizing a limit as the limit of a difference quotient and using our differentiation rules. Thus consider the limits

$$\lim_{x \to 0} \frac{\sin(x)}{x - \pi} \text{ and } \lim_{x \to 0} \frac{\ln(1+x)}{x}.$$

For the first limit we let $f(x) = \sin(x)$ and recognize that $(f(x) - f(\pi))/(x - \pi) = \sin(x)/(x - \pi)$. Thus, $\lim_{x \to \pi} \sin(x)/(x - \pi) = f'(\pi) = -1$. For the second limit, we recognize that the limit is the derivative at 0 of the function $g(x) = \ln(1 + x)$. Thus the value of the limit is $g'(0) = 1$. It is surprising that the limit exists since the numerator and denominator each approach 1.

Today's topic, L'Hôpital's rule, provides a way to evaluate these limits as well as more complicated limits. We will also use this rule to obtain some important information about the behavior of e^x as as x tends to infinity. However, it is important to remember what we have learned about limits. There are some limits where L'Hôpital's rule does not apply and we shall see we need to compute derivatives in order to use l'Hôpital's rule. Computing derivatives requires that we understand limits. Thus we need to be able take limits independently of l'Hôpital's rule if we are to claim that we understand calculus.

1.3 l'Hôpitals' rule

There are many variations of l'Hôpital's rule depending on whether we are taking a limit at a finite value or at infinity and whether the numerator and denominator approach 0 or ∞. We begin with a simple version.

Theorem 1 *Suppose that f and g are functions which are differentiable in an open interval containing a and $f(a) = g(a) = 0$ and assume that $\lim_{x\to a} f'(x)/g'(x)$ exists. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note that as part of the hypothesis that the limit of f'/g' exists, we have that there is small interval (b, c) which contains a and so that $g'(x) \neq 0$ for x in the interval (b, c), except possibly at a. We need this in order for f'/g' to be defined.

We will not prove this theorem.

This situation where $f(a) = g(a) = 0$ will be called the indeterminate form $0/0$.

Example. Evaluate the limits

$$\lim_{x\to 1} \frac{\ln(x)}{x-1}, \qquad \lim_{x\to 0} \frac{\sin(x)}{x+1} \qquad \lim_{x\to 0} \frac{1-\cos(5x)}{x^2}.$$

Solution. We see that $\ln(1) = 1 - 1 = 0$ and thus the first limit is the indeterminate form $0/0$. Using l'Hôpital's rule, we obtain that

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1.$$

The second limit may be evaluated by direct substitution since $1/x$ is continuous at 1.

This limit is not an indeterminate form. The function $\sin(x)/(1+x)$ is continous at 0 and the limit may be evaluated by substitution,

$$\lim_{x\to 0} \frac{\sin(x)}{1+x} = 0.$$

Note that if we try to apply l'Hôpital without checking the hypotheses, we obtain the wrong answer.

The last limit requires some persistence. If $x = 0$, then $1-\cos(5x) = 0$ and $x^2 = 0$. If we apply l'Hôpital's rule once, we obtain the limit of $5\sin(5x)/2x$ which is again the indeterminate form $0/0$. Applying l'Hôpital one more time gives the answer,

$$\lim_{x\to 0} \frac{1-\cos(2x)}{x^2} = \lim_{x\to 0} \frac{5\sin(5x)}{2x} = \lim_{x\to 0} \frac{25\cos(5x)}{2} = 25/2.$$

Or note that we obtain a familiar trig limit after one use of l'Hôpital. ∎

Example. Find the value A so that we may use l'Hopital's rule and give the value of the limit

$$\lim_{x\to 2} \frac{e^{x^2-2x} - A}{x^2-4}.$$

Solution. We have $\lim_{x\to 2}(x^2-4) = 0$. Thus l'Hôpital's rule will apply if the limit of the numerator is also zero,

$$\lim_{x\to 2} e^{x^2-2x} - A = 1 - A = 0.$$

Thus we need $A = 1$ and our limit will be the indeterminate form $0/0$.

Setting $A = 1$ and using l'Hôpital gives

$$\lim_{x\to 2} \frac{e^{x^2-2x} - 1}{x^2-4} = \lim_{x\to 2} \frac{(2x-2)e^{x^2-2x}}{2x} = 1/2.$$

The last limit may be evaluated by substitution since $\frac{(2x-2)e^{x^2-2x}}{2x}$ is continuous at $x = 2$. ∎

1.4 Variations

L'Hôpital's rule continues to apply to limits at infinity, one sided limits, or in the case when f and g approach ∞, rather than 0.

We state two of these variations carefully.

Theorem 2 *If f and g are functions which are defined in an open interval (a, b) which contains c with $\lim_{x\to c} f(x) = g(x) = \infty$. If $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists and*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

If f and g are functions which are defined in an open interval (a, ∞) and $\lim_{x\to\infty} f(x) = g(x) = 0$. If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

In the first part of this theorem, we say that we have the indeterminate form ∞/∞. The result for limits at infinity also holds if we have the indeterminate form ∞/∞. The results hold just as well if we replace ∞ by $-\infty$.

Example. Find the limits

$$\lim_{x\to\infty} \frac{\ln(x)}{x^3}, \qquad \lim_{x\to\infty} \frac{x^3}{e^x}.$$

Solution. We have $\lim_{x\to\infty} \ln(x) = \lim_{x\to\infty} x^3 = \infty$. Thus we have the indeterminate form ∞/∞. Applying l'Hôpital's rule gives that

$$\lim_{x\to\infty} \frac{\ln(x)}{x^3} = \lim_{x\to\infty} \frac{1/x}{3x^2} = \lim_{x\to\infty} \frac{1}{3x^3} = 0.$$

For the limit of x^3/e^x, we have the indeterminate form ∞/∞ and applying l'Hôpital three times gives

$$\lim_{x\to\infty} \frac{x^3}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = 0.$$

∎

This last example can be extended. If we apply l'Hôpital's rule n times we obtain that

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

Thus the exponential function grows faster than any power of x.

Finally, we consider examples where some algebra is needed to apply l'Hôpital's rule.

1.5 The indeterminate form $0 \cdot \infty$.

If we have a functions f and g where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then we have the indeterminate form $0 \cdot \infty$. This can usually be written as the indeterminate form $0/0$ or ∞/∞ and then we may use l'Hôpital's rule.

Example. Find the limit

$$\lim_{x\to 0^+} x\ln(x).$$

Solution. This limit is of the form

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x}$$

which is of the form ∞/∞. Using l'Hôpital's rule gives the limit is of the form

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to\infty} \frac{1/x}{(-1/x^2)} = -\infty.$$

∎

1.6 e

Example. If we invest one dollar at 100% interest, compounded continuously, how much will we have after one year?

Solution. If we divide the year into n periods, each period we multiply our investment by $(1 + 1/n)$. At the end of the year, we have $(1 + 1/n)^n$ dollars. Continuous compounding means we let the period become smaller and smaller and study the behavior. In other words, we take a limit. We may replace the integer n by a real variable x and consider the limit

$$\lim_{x\to\infty}(1 + 1/x)^x.$$

To evaluate the limit we use that $(1 + 1/x)^x = e^{x\ln(1+1/x)}$. The exponent is an indeterminate form of the form $0\cdot\infty$. To evaluate the limit, we rewrite $x\cdot\ln(1+1/x) = \ln(1 + 1/x)/(1/x)$. This gives the indeterminate form $0/0$ and letting $x \to \infty$ gives

$$\lim_{x\to\infty}\frac{\ln(1 + 1/x)}{1/x} = \lim_{x\to\infty}\frac{1}{1 + 1/x}\frac{-1}{x^2}\frac{1}{(-1/x^2)} = 1.$$

Thus

$$\lim_{x\to\infty}(1 + 1/x)^x = \lim_{x\to\infty} e^{x\ln(1+1/x)} = e^{\lim_{x\to\infty} x\ln(1+1/x)} = e.$$

Thus we obtain e dollars at the end of the year. ∎

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