1 Lecture 30: Maximum-minimum problems

- Tests for absolute extrema
- Problem-solving strategy
- Examples

1.1 Tests for absolute extrema

We stated the extreme value theorem which says that every continuous function on a closed interval has an absolute maximum and absolute minimum. We know a maximum exists and we know that it will occur at a critical point or an endpoint. Thus the following procedure will always find the absolute extreme values of a continuous function on a closed interval.

- List the critical points and endpoints.
- Evaluate the function at the critical points and endpoints.
- Choose the smallest and largest values.

There are two more useful tests which do not appear in the text.

Theorem 1 *(The first derivative test for absolute maximum.) Suppose that f is defined and continuous on an interval containing c. If $f'(x) > 0$ if $x > c$ and $f'(x) < 0$ if $x < c$, then c is an absolute minimum.*

Exercise. Restate this test to give a test for an absolute minimum.

A closely related test is the second derivative test. It states that if a function is concave up, then a critical number is an absolute minimum. More precisely, we have:

Theorem 2 *(Second derivative test for absolute extreme values.) Let f be a twice differentiable function defined on an interval and suppose for some c in the interval, $f'(c) = 0$ and that $f''(x) > 0$ for all x in the interval, then f has an absolute minimum at c.*

Proof. To prove this observe that since f'' is positive, then f' is increasing. Since $f'(c) = 0$, we can conclude that $f'(x) > 0$ for $x > c$ and that $f'(x) < 0$ for $x < c$. Thus, c is a local minimum by the first derivative test. ∎

1.2 Strategy

In the examples below, we will follow the following rough guidelines.

1. Read the problem carefully, identify the quantity that we want to make as large or small as possible. This quantity is called the "objective function".
2. Draw a diagram and introduce variables for all quantities from the problem.
3. Write an expression for the objective function. This expression may involve more than one variable.
4. Write the constraint equations. The constraint equation is an equation relating the quantities in the problem. Use the constraint equation to eliminate extra variables from our objective function.
5. Write clearly the function (of one variable) to be optimized and state the domain. The domain may be smaller than the natural domain where the function is defined. These restrictions on the domain may be stated in the problem or may arise because of geometric reality that a negative length is not likely to occur.
6. Find the extreme value of the objective function using one of the tests above. Explain why you know you have found the maximum.
7. Answer the question. Are you to give the location of the maximum, the extreme value, or do you need to compute additional quantities?

1.3 Examples

Example. Suppose the product of two positive numbers is 5. What is the smallest possible value for the sum? The largest possible value?

Solution. (Minimum value) We let a and b be the two numbers. We are told that a and b satisfy the equation $ab = 5$. Our goal is to maximize the objective function $a + b$.

We solve $ab = 5$ to express b in terms of a (Does it matter if we solve for b or a?) which gives $b = 5/a$. Substituting for b, we obtain the objective function:

$$f(a) = a + \frac{5}{a}$$

and we want to find the smallest value of f for a in $(0, \infty)$.

If we compute the derivatives we obtain:

$$f'(a) = 1 - \frac{5}{a^2} \qquad f''(a) = \frac{10}{a^3}.$$

Thus $a = \sqrt{5}$ satisfies $f'(\sqrt{5}) = 0$ and $f''(x) > 0$ for all $x > 0$. We may use the second derivative test for absolute minimum to conclude that $a = \sqrt{5}$ is an absolute minimum on $(0, \infty)$.

The question asks for the minimum value of the sum. This is $f(\sqrt{5}) = 2\sqrt{5}$.

(Maximum value) A sketch the graph of f will indicate that f does not have a maximum value. In fact,

$$\lim_{a \to 0^+} f(a) = \lim_{a \to \infty} f(a) = +\infty.$$

Thus, the sum can be arbitrarily large and does not attain a maximum value. ∎

Example. A piece of wire is cut into two pieces. One piece is bent to form a square and the other to form a circle.

How should we cut the wire into make the area as large as possible? As small as possible? We allow the possibility that one piece is of zero length.

Solution. We let s be the length of the wire and divide the wire into two pieces, $s = x + y$. We use x to form the square and y to form a circle. If the perimeter of the square is x, then the sidelength is $x/4$ and its area is $x^2/16$. The perimeter of the circle is y and if we let r be the radius, we have $y = 2\pi r$ or $r = y/(2\pi)$. The area will be $\pi y^2/(4\pi^2) = y^2/(4\pi)$. Thus the total area is

$$A = x^2/16 + y^2/(4\pi).$$

We need to eliminate one variable and we use the relation $x + y = s$ to do this. It does not matter very much which variable we eliminate. I choose to write $x = s - y$ and this gives the area is

$$A(y) = (s - y)^2/16 + y^2/(4\pi).$$

Since we need both x and y positive (we allow zero), we require that y lie in the interval $[0, s]$. Thus we want to find the maximum and minimum values of A on the interval $[0, s]$.

According the closed interval method, we need to find the critical points of A and test the function A at the endpoints and critical points. The derivative is $A'(y) = (y - s)/8 + y/(2\pi)$. Solving $A'(y) = 0$, gives that $y(1 + 4/\pi) = s$ or $y = \pi s/(4 + \pi)$. This critical is in the interval $[0, s]$. The endpoints are 0 and s, of course. Substituting we obtain

	y	$A(y)$
End point	0	$\frac{s^2}{16}$
Critical point	$\frac{\pi s}{\pi + 4}$	$\frac{s^2}{4(4 + \pi)}$
End point	s	$\frac{s^2}{4\pi}$

We compute the area at the critical point,

$$
\begin{aligned}
A(\pi s/(\pi+4)) &= \frac{1}{16}\left(\frac{4s}{4+\pi}\right)^2 + \frac{1}{4\pi}\left(\frac{\pi s}{4+\pi}\right)^2 \\
&= \left(\frac{s}{4+\pi}\right)^2 (1+\frac{\pi}{4}) = \frac{s^2}{4(4+\pi)}
\end{aligned}
$$

It is clear that the largest area occurs when we use the entire wire to make a circle and the smallest area occurs when we use $s\pi/(\pi+4)$ to make a circle and $4s/(\pi+4)$ to make a square. ∎

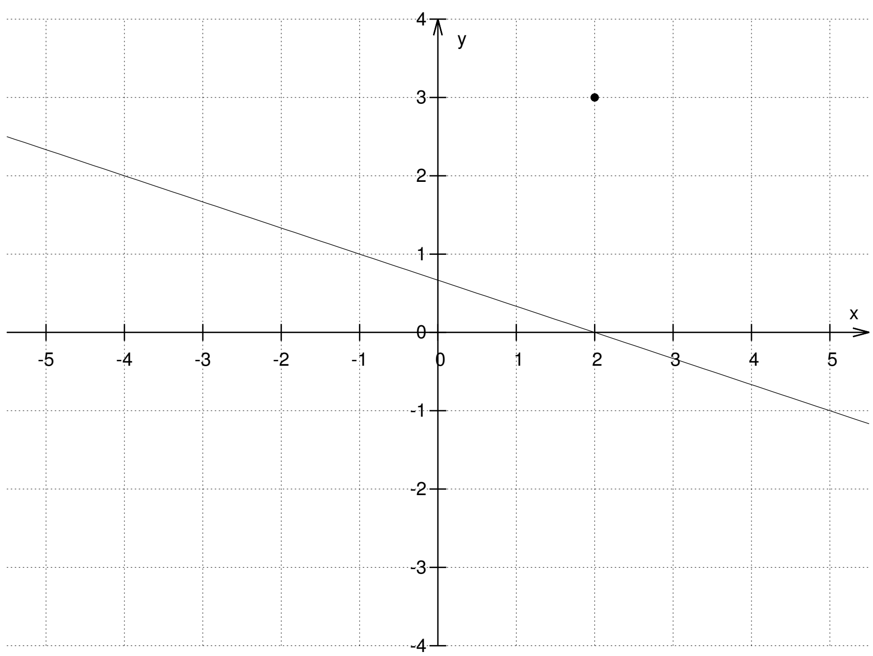
Example. Find the point on the line $x+3y=2$ which is closest to $(2,3)$.

Solution. We let (x,y) be an arbitrary point in the plane and then use the distance formula to write down the distance between (x,y) and $(2,3)$.

$$d = \sqrt{(x-2)^2 + (y-3)^2}.$$

If the point (x,y) is to lie on the line, then we have the relation between the coordinates x and y: $x+3y=2$. We can solve this equation to give $x = 2-3y$ and substituting for x in d, gives

$$d = \sqrt{((2-3y)-2)^2 + (y-3)^2}.$$



A trick that will simplify the calculations below is to realize that the minimum value of d and the minimum value of d^2 occur at the same point. Thus, our goal is find the minimum value of

$$\begin{aligned} f(y) = d^2 &= ((2-3y)-2)^2 + (y-3)^2 \\ &= 9y^2 + y^2 - 6y + 9 \\ &= 10y^2 - 6y + 9 \end{aligned}$$

for y in $(-\infty, \infty)$.

Computing the derivatives, $f'(y) = 20y - 6$ and $f''(y) = 20$. Thus, we can use the second derivative test for absolute extreme values to conclude that the minimum occurs when $y = 3/10$. Since $x = 2 - 3y$, the nearest point will be $(x, y) = (11/10, 3/10)$. ▮

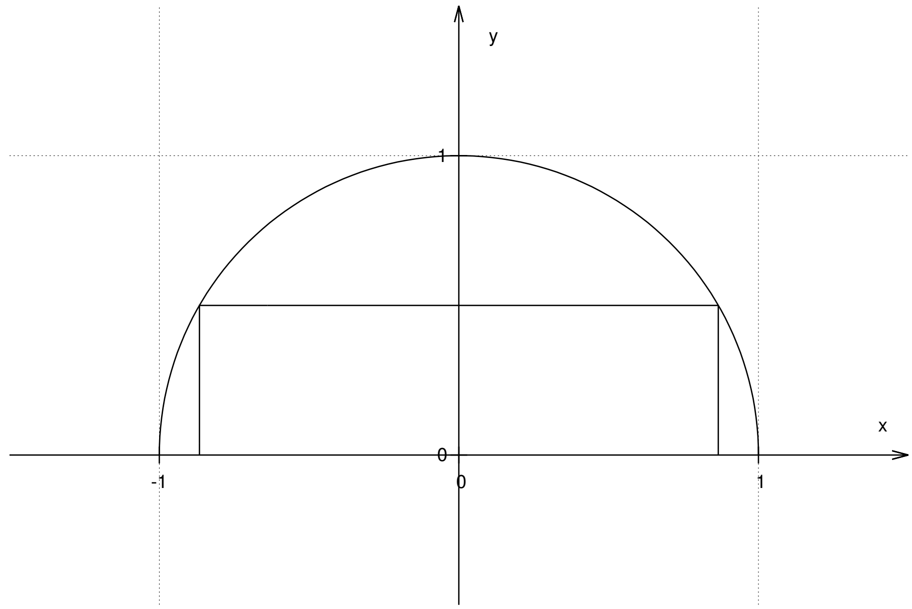
Remark. It is overkill to use calculus to find the minimum value of a parabola. A more appropriate solution is by completing the square:

$$\begin{aligned} 10y^2 - 6y + 9 &= 10(y^2 - \frac{6}{10}y + \frac{9}{100}) - \frac{9 \cdot 10}{100} + 9 \\ &= 10(y - \frac{3}{10})^2 + \frac{81}{100}. \end{aligned}$$

Since a square of a real number is always positive, we conclude that the minimum value occurs when $y = 3/10$.

Some of you may know that the line segment joining a point in the plane to the nearest point on a given line will be perpendicular to the given line. The proof of this fact uses calculus as in the argument above. You may use this fact to check if our answer is correct. ▮

Example. One side of a rectangle rests on the x-axis and two vertices touch the circle $x^2 + y^2 = 1$. Find the largest possible area.



Solution. We suppose that the vertex in the first quadrant is (x, y), then the area of the rectangle is

$$A = 2xy.$$

Because the point (x, y) lies on the circle, we have the relation $x^2 + y^2 = 1$. We may solve this equation to eliminate one of the variables, let us pick y. Then, $y = \sqrt{1 - x^2}$ so that

$$A(x) = 2x\sqrt{1 - x^2}$$

and we need to find the maximum value of the objective function $A(x)$ on the interval $[0, 1]$. The domain is restricted to $[0, 1]$ since the point (x, y) was chosen in the first quadrant.

Thus, we use the procedure for finding the maximum value of a continuous function on a closed interval. We test the endpoints, 0, 1, and the critical number $1/\sqrt{2}$. The maximum occurs at $x = 1/\sqrt{2}$. The largest area is 1. ∎

Second solution. When working with circles, one can also use trigonometric functions. If we let the vertex in the first quadrant be $(\cos(\theta), \sin(\theta))$, then our goal is find the maximum value of

$$A(\theta) = 2\cos(\theta)\sin(\theta).$$

This maximum occurs when $\theta = \pi/4$. ∎

Example. A rectangular poster has area of 6000 cm^2. The poster is to consist of a rectangular printed area in the center with 10 cm margins top and bottom and 6 cm margins left and right. Find the dimensions of the poster with the largest printed area.

Solution. We let x be the width and y the height of the poster. The printed area will have area

$$(x-12)(y-20).$$

The constraint that relates the variables x and y is the condition that the area of the poster (including margins) is 6000 cm^2, $xy = 6000$. If we use this to eliminate y, we find the printed area as a function of x,

$$P(x) = (x-12)(\frac{6000}{x} - 20) = 6000 - \frac{72,000}{x} - 20x + 240.$$

We need $x \geq 12$ and $y \geq 20$ to allow for the margins. If $y \geq 20$, we must have $x \leq 300$. Thus our goal is to find the maximum value of $P(x)$ for x in $[12, 300]$. We know that the maximum value of the continuous function P on the closed interval $[12, 300]$ will be at a critical point or an endpoint of the interval. We compute the derivative

$$P'(x) = \frac{72,000}{x^2} - 20.$$

Solving $P'(x) = 0$ for critical points, we have $x^2 = 3600$ or $x = \pm 60$. Only the positive value is in our domain $[12, 300]$. We check the values of P at the critical point and the endpoints and find

x	$P(x)$
12	0
60	3840
300	0

The maximum printed area will be 3840 cm^2 and will occur when the width is 60 cm and the height is 100 cm. ∎

April 1, 2014