1 Lecture 32: Anti-differentiation.

- What is an anti-derivative?
- Computing anti-derivatives.
- How many anti-derivatives does a function have?
- Finding position given acceleration. An application of anti-derivatives.

1.1 What is an anti-derivative?

If we know the velocity of an object, it seems likely that we ought to be able to recover how far and in what direction the object has traveled. If we also know where the object started, then we can determine position of the object from its velocity and its starting position. Since velocity is the derivative of position, to solve this problem of recovering position, we need to be able to recover a function from its derivative. To ease this process, we introduce a name for "a function whose derivative is f".

Definition. An anti-derivative of f is a function whose derivative is f. Thus, if F is an anti-derivative of f, then $F' = f$.

The process of finding a function from its derivative is called *anti-differentiation.* As the name implies, anti-differentiation is an inverse operation to differentiation. If we have a table giving functions in the left-hand column and the derivatives on the right, then we can find an anti-derivative by searching for a function in the right-hand column and then anti-derivative will be to the left.

Example. Show that $g(x) = x^2 + \sin(2x) + 2$ is an anti-derivative of $f(x) = 2x + 2\cos(x)$. Find another anti-derivative of f.

Solution. We need to compute the derivative of g and see if we obtain the function f.

The derivative is $g'(x) = 2x + 2\cos(2x)$ which is the function $f(x)$.

Another anti-derivative is $g(x) + 99$. ∎

Remark. This example illustrates that if we have found an anti-derivative, then we can check our answer by differentiating. In general, finding anti-derivatives can be tricky. ∎

1.2 Finding anti-derivatives.

In the first part of this course, we have computed the derivatives of a number of functions. We summarize sum of the rules in the table below.

Function	Derivative
x^r	rx^{r-1}
e^x	e^x
$\ln(x)$	$1/x$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\arctan(x)$	$\dfrac{1}{1+x^2}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\operatorname{arcsec}(x)$	$\dfrac{1}{\|x\|\sqrt{x^2-1}}$
$f(x)+g(x)$	$f'(x)+g'(x)$
$cf(x)$	$cf'(x)$

Each of these entries can be rewritten to give a rule for anti-differentiation.

Anti-derivatives of sums and multiples. If F is an anti-derivative of f and G is an anti-derivative of g, then $F+G$ will be an anti-derivative of $f+g$ and cF will be an anti-derivative of cf provided c is a constant.

Anti-derivatives of powers. Let s be a real number, then the power rule tells us

$$\frac{d}{dx}x^s = sx^{s-1},$$

we can divide both sides by the constant s and conclude that

$$\frac{x^s}{s}$$

is an anti-derivative of x^{s-1}. This rule becomes more useful if we replace s by $r+1$ and $s-1$ by r which gives: An anti-derivative of x^r is $\frac{x^{r+1}}{r+1}$. Of course we need for $r+1$ to be non-zero or $r \neq -1$.

Rewriting the entries in the table above gives:

Function	An anti-derivative
x^r, $r \neq 1$	$\frac{x^{r+1}}{r+1}$, $r \neq -1$
$1/x$	$\ln(\lvert x \rvert)$
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\dfrac{1}{1+x^2}$	$\arctan(x)$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$\dfrac{1}{\lvert x \rvert\sqrt{x^2-1}}$	$\mathrm{arcsec}(x)$
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(X)$

In this table, we assume that c is a constant, F is an anti-derivative of f and G is an anti-derivative of g. We also point out that it is common to use the symbol

$$\int f(x)\,dx$$

for an anti-derivative of f. Thus, the equation

$$\int x\,dx = \frac{x^2}{2} + C$$

means that for any constant C, $x^2/2 + C$ is an anti-derivative of f. The only new fact in the above table is the anti-derivative of $1/x$. We know that $\frac{d}{dx}\ln(x) = 1/x$ if $x > 0$. If $x < 0$, then $|x| = -x$ and

$$\frac{d}{dx}\ln(-x) = -1 \cdot \frac{1}{-x} = 1/x$$

by the chain rule.

The alert student will observe that we have not talked about using the product rule or chain rule as rules for anti-differentiation. This will be taken care of later.

Example. Find anti-derivatives of the following functions:

$$\frac{1}{x}, \qquad x^2 + \frac{2}{x^2}, \qquad 3\sin x + 4\cos x, \qquad x\cos(x^2)$$

Solution. If we try to use the rule for an anti-derivative of a power, we find that the anti-derivative of $1/x = x^{-1}$ is

$$\frac{x^{-1+1}}{-1+1}.$$

This involves dividing by 0 and hence is non-sense. This illustrates why we have the restriction $r \neq 1$ for anti-derivatives of powers. An anti-derivative of $1/x$ is $\ln|x|$.

In the second example, we rewrite $2/x^2$ as $2x^{-2}$ and use the power rule and the rules for anti-derivatives of sums and multiples to obtain that an anti-derivative of $x^2 + 2x^{-2}$ is

$$\frac{x^3}{3} + 2\frac{x^{-1}}{-2+1} = \frac{x^3}{3} - \frac{2}{x}.$$

For the third example, we use the anti-derivatives of sin and cos from the table and that the rules for anti-derivatives of sums and multiples. The anti-derivative of $3\sin x + 4\cos x$ is

$$-3\cos x + 4\sin x$$

We do not have a systematic way to solve the last problem, but if we are told that the anti-derivative of $x\cos(x^2)$ is

$$\frac{1}{2}\sin(x^2)$$

then one may easily check whether or not this is correct by computing the derivative of the candidate for the anti-derivative. ∎

Exercise. Check the answers in the above example by computing the derivative of the anti-derivative to see if we recover the original function.

1.3 How many anti-derivatives does a function have?

The following theorem appears in section 4.3 of Rogawski.

Theorem 1 *If $f'(x) = 0$ for all x in an open interval (a, b), then f is constant.*

Proof. If we choose x and y with $a < x < y < b$ and apply the mean value theorem on the interval $[x, y]$, we have $f(y) - f(x) = f'(c)(y - x)$ for some c in (x, y). But $f'(c) = 0$ so $f(x) = f(y)$, so f is constant. ∎

Corollary 2 *If F and G are anti-derivatives on the open interval (a, b), then $F = G + C$ for some constant C.*

The corollary follows immediately from Theorem 1.

Example. Find all the anti-derivatives of $\frac{1}{x^3}$.

Solution. According to the rule for finding an anti-derivative of a power function, the anti-derivative of x^{-3} is $\frac{-1}{2}x^{-2}$. Notice that this function is defined $(-\infty, 0) \cup (0, \infty)$. Any anti-derivative of x^{-3} will differ from $-1/(2x^2)$ by a constant on the interval $(-\infty, 0)$ and perhaps by another constant on the interval $(0, \infty)$, thus the most general anti-derivative of $1/x^3$ is

$$F(x) = \begin{cases} \frac{-1}{2x^2} + C_1, & x > 0 \\ \frac{-1}{2x^2} + C_2, & x < 0 \end{cases}$$

∎

Example. Find an anti-derivative F of $f(x) = x^2$ which satisfies $f(2) = 1$.

Solution. The most general anti-derivative of x^2 is

$$F(x) = \frac{1}{3}x^3 + C.$$

If we want $F(2) = 1$, then C should satisfy the equation

$$\frac{8}{3} + C = 1.$$

The solution of this equation is $C = -5/3$. Thus the function F is

$$F(x) = \frac{1}{3}x^3 - \frac{5}{3}.$$

∎

Example. Find a function which satisfies $f''(x) = \cos x$, $f(0) = 1$ and $f'(0) = 2$.

Solution. We begin by taking the anti-derivative twice. If $f''(x) = \cos x$, then $f'(x) = \sin(x) + A$ and $f(x) = -\cos x + Ax + B$. The conditions on $f(0)$ and $f'(0)$ give the equations:

$$\begin{aligned} f'(0) = A &= 2 \\ f(0) = -1 + B &= 1 \end{aligned}$$

It is easy to see that the solution is $A = 2$ and $B = 2$ and thus

$$f(x) = -\cos x + 2x + 2.$$

∎

Exercise. Check the answer to the previous example.

1.4 Recovering position from acceleration

If we throw a ball up in the air and neglect air resistance, then the acceleration of the object in the vertical direction is constant. This constant is usually written g where g is a positive number and the minus sign indicates that the force of gravity tends to pull the ball down, or in the negative direction. The value g is -32 feet/second^2 or -9.8 meters/second^2 in the metric system. Since the acceleration is the second derivative of the height function, the problem of finding the height function is a problem of finding a function $h(t)$ which satisfies $h''(t) = g$. Using the rules for finding anti-derivatives, we know that the velocity at time t is

$$h'(t) = gt + A$$

and the height is

$$h(t) = \frac{1}{2}gt^2 + At + B.$$

Since the constant A is the velocity at time $t = 0$, we usually replace A by v_0 and since the constant B is the height at time 0, we replace B by h_0. Thus, the height at time t is

$$h(t) = -\frac{1}{2}gt^2 + v_0t + h_0.$$

Example. Suppose a ball is thrown up from the ground, find the time t when it returns to the ground. Your answer will depend on g and v_0.

Suppose that the ball stays in the air for 4 seconds. Find the value of the initial velocity in meters/second.

Solution. Since the ball starts on the ground, its height at time $t = 0$ is $h_0 = 0$. The height at time t is $h(t) = \frac{1}{2}gt^2 + v_0t = t(v_0 + \frac{1}{2}gt)$. The two roots of $h(t) = 0$ are $t = 0$ and $t = -2v_0/g$. The first root $t = 0$ corresponds to the time ball is thrown and the root $t = -2v_0/g$ gives the time when the ball returns to the ground.

If the ball stays aloft for $t = 4$ seconds, we know that

$$t = -\frac{2v_0}{g}$$

and hence

$$v_0 = -\frac{1}{2}tg.$$

Substituting the value $g = -9.8$ meters/second^2 and $t = 4$ seconds gives

$$v_0 = \frac{1}{2}(4 \text{ seconds})(9.8\frac{\text{meters}}{\text{second}^2}) = 19.6 \text{ meters/second}^2.$$

I believe this works out to about 44 miles per hour. ∎

Remark. This problem is interesting because it provides a simple way to measure the velocity of a thrown ball. If one throws a ball, it is not easy to measure the velocity–unless you have a radar gun. However, if you throw the ball up in the air, one can measure the time before it returns to the ground with a stop-watch. Using this value and the result of the above problem, one can compute the initial velocity. ∎

Example. Suppose that an object starts at the origin and its initial position is 0. The velocity at time t is $v(t) = 2\sin(t)$ meters/second. Find the position at time t.

Solution. To find position, we need the anti-derivative of velocity. Since $\sin(t)$ is defined on the real line, the most general anti-derivative is

$$p(t) = -2\cos(t) + C.$$

To have $p(0) = 0$, we need $C = 2$. Thus the answer is

$$p(t) = 2 - 2\cos(t).$$

∎

1.5 Further examples

Example. Suppose that a ball is thrown up from the top of a 100 meter building with an initial velocity of 20 meters/second. Find the velocity of the ball at the instant it hits the ground.

Solution. We are given that the initial height is $h_0 = 100$ and the initial velocity v_0 is 20 meters/second. Thus the height in meters t seconds after the ball is thrown is

$$h(t) = -4.9t^2 + 20t + 100.$$

Here, we take $h = 0$ to correspond to ground level. Thus $t = 0$, $h(0) = 100$, the height of the building in meters.

We solve $h(t) =$ using the quadratic formula and obtain that the ball reaches the ground when

$$t = \frac{-20 \pm \sqrt{20^2 + 4 \cdot 4.9 \cdot 100}}{9.8}$$

Evaluating this expression, we find t is approximately +6.78 seconds or -2.7 seconds. Since we are not interested in what happens before the ball is thrown, we discard the negative answer and conclude the ball lands approximately 6.78 seconds after it was thrown.

The velocity of the ball at this time is approximately

$$h'(6.78) = -46.48 \text{ meters/second}.$$

As expected, the velocity is negative because the ball is traveling down when it hits the ground. ∎