1 Lecture 36: Integrals.

- The definition of the integral.
- Properties of the integral: linearity.
- Order properties.
- Using the interpretation as area to evaluate simple integrals.

1.1 The definition of the integral

In the previous lecture we discussed approximating the area of a region using a sum of thin rectangles. If the limits of the right endpoint, left endpoint and midpoint approximations to the area

$$\lim_{N\to\infty} R_N, \lim_{N\to\infty} M_N, \lim_{N\to\infty} L_N$$

exist and are equal, then this number is a reasonable candidate for the area.

We consider similar sums, called Riemann sums to define the integral. Consider a continuous function f defined on an interval $[a, b]$ and let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a *partition*, a collection of points that subdivide the interval into n subintervals. For each i, choose a sample point c_i in $[x_{i-1}, x_i]$. With this notation, the Riemann sum for f depending on P and C is

$$R(f, P, C) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}).$$

We will sometimes write $\Delta x_i = x_i - x_{i-1}$ to be the change in x on the ith interval. Finally, we define the norm of the partition by $\|P\| = \max\{|x_i - x_{i-1}| : i = 1, \ldots, n\}$. The *defnite integral of the function f over $[a, b]$* is the limit of the Riemann sums,

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to\infty} R(f, P, C)$$

provided the limit exists. Note that since the sub-intervals may be of different lengths, it is no longer enough to send the number of intervals to infinity. We have to make sure the width of each interval goes to zero.

The following result gives a condition that guarantees that the integral is defined.

Theorem 1 *If f is continuous on $[a, b]$, then*

$$\int_a^b f(x)\,dx$$

exists.

Example. Let $f(x) = 1/x$ on $[1, 3]$ and compute the Riemann sum with $P = \{1, 5/4, 3/2, 2, 3\}$ and $C = \{1, 3/2, 7/4, 5/2\}$. Sketch the area that this sum represents.

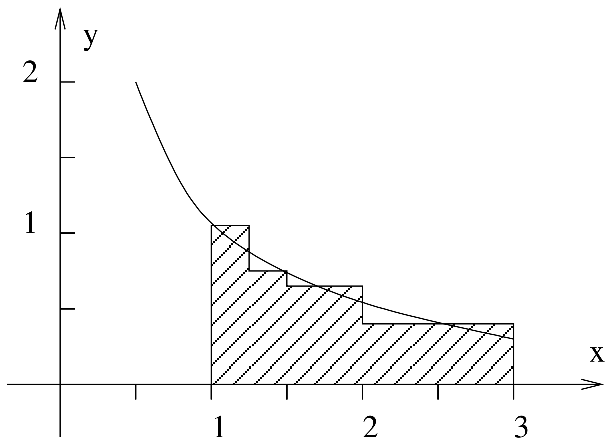


Figure 1: A Riemann sum

Solution. The computation is summarized in the table below.

Interval	Δx	$f(c_i)$	Area
$[1, 5/4]$	1/4	1	1/4
$[5/4, 3/2]$	1/4	2/3	1/6
$[3/2, 2]$	1/2	4/7	2/7
$[2, 3]$	1	2/5	2/5
Total			1.1024

This is the shaded area in the Figure 1. ∎

Note that if $f \geq 0$, then the integral may be interpreted as the area of the region $\{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$. If f is negative, the integral is the negative of the area between the graph of f and the line $y = 0$. If A is the (positive) area of the region in Figure 2, then we have $\int_a^b f(x) = -A$.

If f is of variable sign, then the integral will be

$$\int_a^b f = A_+ - A_-$$

where A_+ is the area above the x-axis and A_- is the area below the x-axis.

Example. Eventually, we will learn that

$$\int_0^{2\pi} \sin x \, dx = 0.$$

If we examine the graph of $\sin(x)$ for x in $[0, \pi]$, it appears to enclose equal areas above and below the x-axis. Thus, this result is plausible.

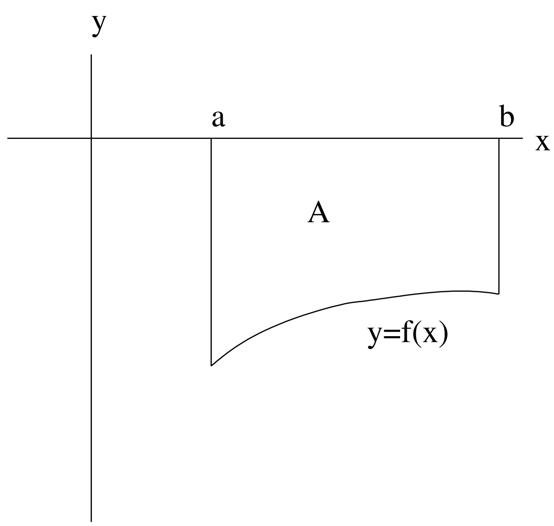
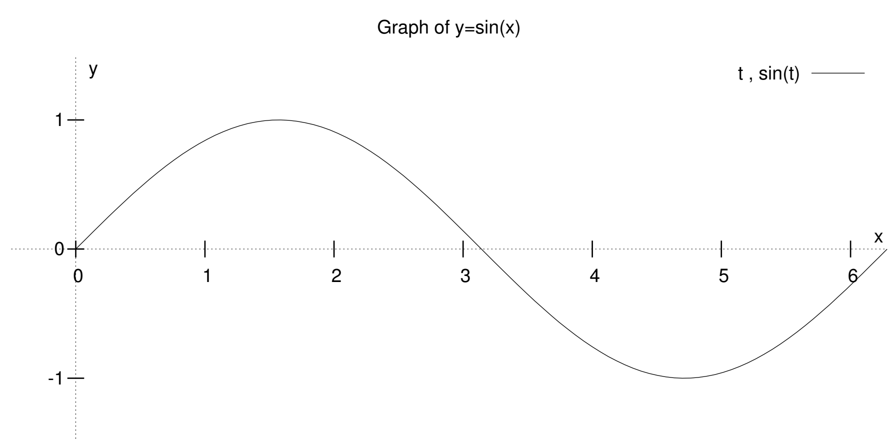


Figure 2: If the graph of f lies below the x-axis, the area is negative.



Finally, we define the integral when $b \leq a$ by

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \qquad \int_a^a f(x)\,dx = 0.$$

1.2 Properties

We list several basic properties of the definite integral. Suppose all of the integrals below exist, and c is a constant:

- $\int_a^b c\,dx = c(b-a)$
- $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

The first of these properties is a generalization of the formula for the area of a rectangle. The second and third state that the integral is linear. We will not prove these statements. The fourth states that if we take the union of two disjoint regions, the area of the union is the sum of the areas of each piece. The last is a restatement of part of our definition of the integral.

Example. Find $\int_2^3 (2+3x)\,dx$ given that $\int_0^a x\,dx = a^2/2$.

Solution. We use linearity to write

$$\int_2^3 (2+3x)\,dx = \int_2^3 2\,dx + 3\int_2^3 x\,dx = 2 + 3\int_0^3 x\,dx - 3\int_0^2 x\,dx = 2 + 27/2 - 6 = 19/2.$$

∎

1.3 Order Properties

In this section, we list several order properties of the integral:

Theorem 2 *Suppose the integrals below exist and $a < b$.*
If $f(x) \geq 0$ for all x in $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.
If $f(x) \geq g(x)$ for all x in $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
If $m \leq f(x) \leq M$ for all x in $[a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Proof. The first statement follows because every Riemann sum for f is non-negative.
The second and third are easy consequences of the first. ∎

Example. Find a number c so that

$$\int_1^c \frac{1}{x}\,dx \geq 1.$$

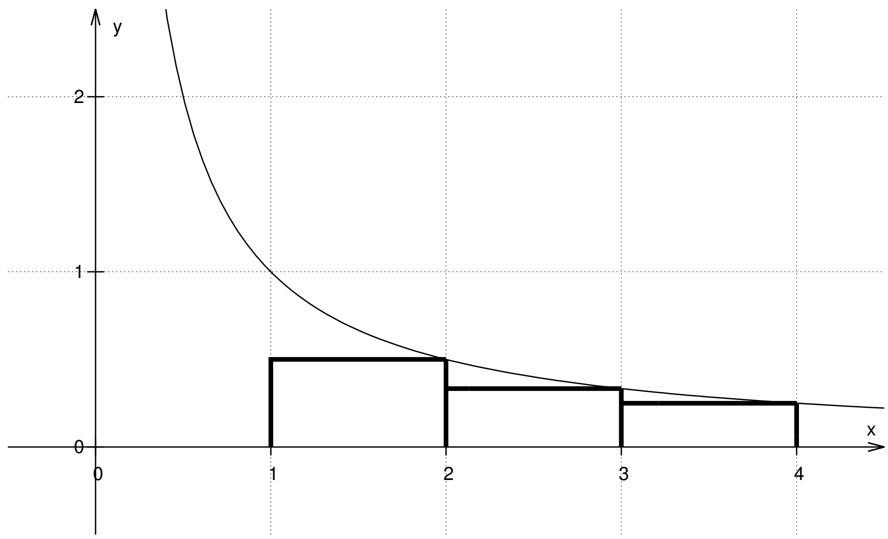


Figure 3: Riemann sum for $f(x) = 1/x$.

Solution. If we consider regions smaller than the area, we find that the Riemann sum using the partition $P = \{1, 2, 3, 4\}$ and the right endpoints as sample points gives

$$\int_1^4 \frac{1}{x}\,dx \geq (\frac{1}{2} + \frac{1}{3} + \frac{1}{4}).$$

Thus $c = 4$ will work.

∎

Exercise. Can you find a smaller value of c? What is the smallest value of c for which

$$\int_1^c \frac{1}{x}\,dx \geq 1.$$

Example. Use well-known formulae for area to find the following integrals:

$$\int_{-3}^{3} \sqrt{9 - x^2}\,dx \qquad \int_{3}^{-1} 2x\,dx. \qquad \int_{1/\sqrt{2}}^{1} \sqrt{1 - x^2}\,dx.$$

Solution. The first area represents the area of a circle of radius 3. Its value is $9\pi/2$.

The second integral can be rewritten as the difference of the areas of two triangles.

$$\int_{3}^{-1} 2x\,dx = -\int_{-1}^{0} 2x\,dx - \int_{0}^{3} dx = 1 - 9 = -8.$$

The third integral can be viewed as the area of a sector of radius π with a triangle removed. The value is $\pi/8 - 1/4$. ∎

Remark. A careful student might complain that this exercise requires circular reasoning. We use the integral to give a careful definition of area, but then we assume that we know about area to evaluate integrals. One of the goals of this course is to find a way to calculate integrals (areas) without assuming that we already know the answer. Until we achieve this goal, the above exercises are a good way to gain familiarity with the properties of the integral. ∎

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