# 1 Lecture 37: The fundamental theorems of calculus.

- The fundamental theorems of calculus.
- Evaluating definite integrals.
- The indefinite integral-a new name for anti-derivative.

Today we provide the connection between the two main ideas of the course. The integral and the derivative.

**Theorem 1** *(FTC I) Suppose $f$ is a continuous function on $[a, b]$. If $F$ is an anti-derivative of $f$, then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

*Example.* Compute

$$\int_0^3 x^3\,dx.$$

We give an idea of the proof.

*Proof.* We let $F$ be an anti-derivative of $f$ and let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$. We will express the change of $F$, $F(b) - F(a)$, as a Riemann sum for this partition. Letting the size of the largest interval in the partition tend to zero, we obtain the integral is equal to the change in $F$.

We begin by writing

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \ldots + F(x_i) - F(x_{i-1}) + \ldots + F(x_1) - F(x_0).$$

We recall that $F$ is an anti-derivative of $f$ and apply the mean value theorem on each interval $[x_{i-1}, x_i]$ and find a value $c_i$ so that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$. Thus, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}).$$

Since the right-hand side is a Riemann sum for the integral, we may let the width of the largest subinterval tend to zero and obtain

$$F(b) - F(a) = \int_a^b f(s)\,ds.$$

∎

## 1.1 Indefinite integrals.

We use the symbol

$$\int f(x)\,dx$$

to denote the indefinite integral or anti-derivative of $f$. This should include a constant $C$ to indicate that the choice of indefinite integral involves fixing an arbitrary constant.

The indefinite integral is a function. The definite integral is a number. According FTC I, we can find the (numerical) value of a definite integral by evaluating the indefinite integral at the endpoints of the integral. Since this procedure happens so often, we have a special notation for this operation.

$$F(x)|_{x=a}^{b} = F(b) - F(a).$$

*Example.* Find

$$xa|_{x=a}^{b} \qquad \text{and} \qquad xa|_{a=x}^{y}$$

*Solution.*

$$ba - a^2 \qquad xy - x^2$$

∎

*Example.* Verify

$$\int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + C.$$

*Solution.* According to the definition of anti-derivative, we need to see if

$$\frac{d}{dx}\frac{1}{2}\sin(x^2) = x\cos(x^2).$$

This holds, by the chain rule. ∎

## 1.2 Computing integrals.

The main use of FTC I is to simplify the evaluation of integrals.

We give a few examples.

*Example.* a) Compute

$$\int_0^{\pi} \sin(x)\,dx.$$

b) Compute

$$\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx.$$

c) Find

$$\int_0^1 \frac{1}{1+x^2}\,dx.$$

*Solution.* a) Since $\frac{d}{dx}(-\cos(x)) = \sin(x)$, we have $-\cos(x)$ is an anti-derivative of $\sin(x)$. Using the second part of the fundamental theorem of calculus gives,

$$\int_0^\pi \sin(x)\,dx = -\cos(x)|_{x=0}^\pi = 2.$$

b) We first find an anti-derivative. As the indefinite integral is linear, we write

$$\int \frac{2x^2+1}{\sqrt{x}}\,dx = \int 2x^{3/2} + x^{-1/2}\,dx \quad = \quad 2\int x^{3/2}\,dx + \int x^{-1/2}\,dx = \frac{4}{5}x^{5/2} + 2x^{1/2} + C.$$

With this anti-derivative, we may then use FTC I to find

$$\begin{aligned}\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx &= \left.\frac{4}{5}x^{5/2} + 2x^{1/2}\right|_{x=1}^4 \\ &= \frac{4}{5}4^{5/2} + 24^{1/2} - (\frac{4}{5} + 2) \\ &= 128/5 + 20/5 - (4/5 + 10/5) \\ &= 134/5.\end{aligned}$$

c) We recall that $\arctan(x)$ is an anti-derivative of $1/(1+x^2)$ and thus

$$\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(x)|_{x=0}^1 = \arctan(1) = \pi/4.$$

∎

*Example.* Find

$$\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx.$$

*Solution.* We recognize that $\sin(x^2)$ is an anti-derivative of $2x\cos(x^2)$,

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

Thus,

$$\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx = \sin(x^2)\Big|_{x=0}^{\sqrt{\pi}} = 0 - 0.$$

∎

Note that we needed a certain amount of luck to find this anti-derivative. One of the goals in the future is to learn techniques for finding anti-derivatives in general.

Here, is a more involved example that illustrates the progress we have made.

*Example.* Find

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \sin(k/n).$$

*Solution.* We recognize that

$$\frac{1}{n}\sum_{k=1}^{n}\sin(k/n)$$

is a Riemann sum for an integral. The points $x_k$, $k = 0, \ldots, n$ divide the interval $[0, 1]$ into $n$ equal sub-intervals of length $1/n$. Thus, we may write the limit as an integral

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin(k/n) = \int_0^1 \sin(x)\,dx.$$

To evaluate the resulting integral, we use FTC I. An anti-derivative of $\sin(x)$ is $-\cos(x)$, thus

$$\int_0^1 \sin(x)\,dx = -\cos(x)\big|_{x=0}^{1} = 1 - \cos(1).$$

∎

April 18, 2014