1 Lecture 39: The substitution rule.

- Recall the chain rule and restate as the substitution rule.
- u-substitution, bookkeeping for integrals.
- Definite integrals, changing limits.
- Symmetry-integrating even and odd functions.

1.1 The substitution rule.

Recall the chain rule: If $F' = f$ and g is differentiable, then

$$(F \circ g)'(x) = F'(g(x))g'(x).$$

We can restate this as:

The substitution rule. If F is an anti-derivative of f and g is a differentiable function, then $F \circ g(x)$ is an anti-derivative of $(f \circ g)(x)g'(x)$. In other words,

$$F \circ g(x) = \int f(g(x))g'(x)\, dx.$$

1.2 u-substitution

The Leibniz notation provides a convenient way to keep track of the substitution rule. We let

$$u = g(x), \qquad du = g'(x)dx. \tag{1}$$

To evaluate the indefinite integral

$$\int f(g(x))g'(x)\, dx$$

set $u = g(x)$ and then $du = g'(x)dx$ making these substitutions gives

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du = F(u) = F(g(x)) + C$$

where F is an anti-derivative for f. In a definite integral, we need to also change the limits when $x = a$, then $u = g(a)$ and when $x = b$, $u = g(b)$. Thus, we have

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

An example will illustrate how we use this procedure.

Example. Find

$$\int 2x \sin(x^2)\, dx.$$

Solution. Set $u = x^2$ and then $du = 2xdx$. Making the substitutions as in (1) gives

$$\int 2x \sin(x^2)\, dx = \int \sin u\, du = \cos u + C = \cos(x^2) + C.$$

∎

Exercise. Check our answer by differentiating.

Below is a slightly more interesting example. In this example, we do not find exactly the derivative of $u = g(x)$ hiding in the integral. However, we may multiply the equation $du = g'(x)dx$ by a constant and still use this method.

Example. Find

$$\int \frac{1}{(1-2x)^2}\, dx.$$

Solution. In this example, we only need to substitute by the linear function $u = 1 - 2x$ and then $du = (-2)dx$. In this case, we need to divide by -2 to obtain $\frac{-1}{2} du = dx$. Then we obtain,

$$\int \frac{1}{(1-2x)^2}\, dx = \frac{-1}{2} \int \frac{1}{u^2}\, du = \frac{1}{2} u^{-1} = \frac{1}{2} \frac{1}{1-2x} + C.$$

∎

This works because if $u = g(x)$ and $v = cg(x)$, then we have $dv = c\, du = cg'(x)\, dx$ by the constant multiple rule for differentiation.

Example. Try the substitution $u = \sin(x)$ in the integral

$$\int \sin(x)\, dx.$$

Solution. If $u = \sin(x)$, then $du = \cos(x)\, dx$ or $dx = \frac{1}{\cos(x)}\, du$. Thus we obtain

$$\int \sin(x)\, dx = \int \frac{u}{\cos(x)}\, du.$$

To evaluate this integral, we would need additional work to eliminate the x. Of course, this is not the right away to evaluate this integral since

$$\int \sin(x)\, dx = -\cos(x) + C.$$

For now, we will only multiply the equation relating dx and du by constants. ∎

Example. Find the integral

$$\int \sin(x)\ \cos(x)\,dx$$

Solution. If we set $u = \sin(x)$, then $du = \cos(x)\,dx$ and we have

$$\int \sin(x)\ \cos(x)\,dx = \int u\,dx = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C.$$

If we set $u = \cos(x)$, then $du = -\sin(x)\,dx$ and we have

$$\int \sin(x)\ \cos(x)\,dx = -\int u\,dx = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2(x) + C.$$

Check these answers. Explain why we have found two different answers. ∎

1.3 Definite integrals.

To evaluate definite integrals, we have a choice. We may change the limits as described above. Another approach is to separate the steps of finding the anti-derivative and evaluating the anti-derivative. In this approach, we would use substitution to find the indefinite integral and then evaluate to find the definite integral.

We give a simple example where we change limits.

Example. Find

$$\int_1^4 \sqrt{2x+1}\,dx.$$

Solution. Set $u = 2x + 1$ and then $du = 2dx$. If $x = 1$, then $u = 3$ and if $x = 4$, then $u = 9$. Thus,

$$\begin{aligned}\int_1^4 \sqrt{2x+1}\,dx &= \frac{1}{2}\int_3^9 u^{1/2}\,du \\ &= \frac{1}{2}\frac{2}{3}\,u^{2/3}\Big|_3^9 \\ &= \frac{1}{3}(9^{3/2} - 3^{3/2}) = 9 - \sqrt{3}.\end{aligned}$$

∎

Here is a solution following the strategy of separating the steps.

Solution. Set $u = 2x + 1$ and then $du = 2dx$. If $x = 1$, then $u = 3$ and if $x = 4$, then $u = 9$. Thus,

$$\begin{aligned} \int \sqrt{2x+1}\, dx &= \frac{1}{2} \int u^{1/2}\, du \\ &= \frac{1}{2}\frac{2}{3} u^{3/2} + C \\ &= \frac{1}{3}(2x+1)^{3/2} + C. \end{aligned}$$

Now that we have the anti-derivative, we may use the Fundamental Theorem of Calculus to obtain

$$\int_1^4 \sqrt{2x+1}\, dx = \frac{1}{3}(2x+1)^{3/2}\Big|_1^4 = \frac{1}{3}(9^{3/2} - 3^{3/2}) = 9\sqrt{3}.$$

∎

Finally, we give an example where a bit more algebra is needed.

Example. Find the anti-derivative

$$\int x\sqrt{2x+1}\, dx.$$

Solution. Again, we substitute $u = 2x + 1$ and $du = 2dx$ or $dx = frac12du$ but this leaves an x. We solve $u = 2x + 1$ to express $x = \frac{1}{2}(u - 1)$. Making the substitutions, we have

$$\int x\sqrt{2x+1}\, dx = \int \frac{1}{2}(u-1)u^{1/2}\frac{1}{2}du = \frac{1}{4}\int (u^{3/2} - u^{1/2})\, du.$$

Taking the anti-derivative and then replacing u by $2x + 1$ gives

$$\frac{1}{4}\int (u^{3/2} - u^{1/2})\, du = \frac{2}{20}u^{5/2} - \frac{2}{12}u^{3/2} + C.$$

And replacing u by $2x + 1$ gives

$$\int x\sqrt{2x+1}\, dx == \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C.$$

∎

1.4 Quadratic expressions

We recall several anti-differentiation formulae involving inverse trig functions.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C, \qquad \int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

and

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + C.$$

Often we can reduce other integrals involving quadratic expressions to one of these by a substitution.

Example. Find the indefinite integrals

$$\int \frac{1}{x^2+4}\,dx, \qquad \int \frac{1}{4x^2+9}\,dx.$$

Solution. In the first example, let $x = 2u$, $dx = 2du$. With this we have a common factor in the denominator and obtain

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4u^2+4}\,2du = \frac{2}{4}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u)+C = \frac{1}{2}\arctan(2x)+C.$$

Check your answer by differentiating!!!

For the second example, we would like a common factor in the denominator. We may write $4x^2+9 = 9(\frac{4}{9}x^2+1)$. Thus if we substitute $u = 2x/3$ we will obtain a familiar integral.

$$\int \frac{1}{9+4x^2} = \int \frac{1}{9((2x/3)^2+1)}\,dx$$

Now substituting $u = 2x/3$ or $du = \frac{2}{3}dx$, we obtain

$$\int \frac{1}{9((2x/3)^2+1)}\,dx = \frac{1}{9}\int \frac{1}{u^2+1)}\frac{3}{2}du = \frac{1}{6}\arctan(u) + C = \frac{1}{6}\arctan(2x/3) + C.$$

∎

Example. Complete the square to find

$$\int \frac{1}{\sqrt{2x-x^2}}\,dx.$$

Solution. If we complete the square, we may write $2x - x^2 = 1 - (x^2 - 2x + 1) = 1 - (x-1)^2$. Thus, we have

$$\int \frac{1}{\sqrt{2x - x^2}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx.$$

If we substitute $u = x - 1$, $du = dx$, we obtain

$$\int \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,dx = \arcsin(u) + C = \arcsin(x-1) + C.$$

∎

1.5 Further topics, symmetry

The substitution $u = -x$ gives

$$\int_0^a f(x)\,dx = \int_{-a}^0 f(-u)\,du.$$

If f is odd, or even, this simplifies further.

A function is even if $f(-x) = f(x)$. For even functions we have

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

A function is odd if $f(-x) = -f(x)$ and for odd functions,

$$\int_{-a}^a f(x)\,dx = 0.$$

Example. Find

$$\int_{-2}^2 x^3 + x^2 + x + 2\,dx \qquad \int_{-1}^1 x^{101}\sin(x^{100})\,dx \qquad \int_{-10}^{11} x\,dx.$$