1 Lecture 41: Area between curves

- Expressing area as a limit of Riemann sums.
- Examples.
- Choosing axes

1.1 Area as a limit of Riemann sums

We recall that the integral is a limit of Riemann sums.

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{m} f(c_i)(x_i - x_{i-1}).$$

Let f and g be two functions with $f(x) \le g(x)$. We will use an integral to find the area of the region in the plane $R = \{(x,y) : a \le x \le b,\ f(x) \le y \le g(x)\}$. Let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ be a partition of the interval $[a,b]$ and let $\{c_i\}_{i=1}^n$ be a set of sample points with $c_i \in [x_{i-1}, x_i]$. The sum

$$\sum_{i=1}^{n} (g(c_i) - f(c_i))(x_i - x_{i-1})$$

gives an approximation of the area R. As the width of the largest subinterval becomes small, the sum becomes a better approximation of the area and approaches an integral. We obtain

$$\text{Area of } R = \int_a^b (g(x) - f(x))\,dx. \tag{1}$$

1.2 Finding area

To use the expression (1), there are a couple of details that we need to take care of.

- As always, begin by making a sketch of the region.
- Find the endpoints of the interval of interest.
- Determine which function lies above. If the graphs cross, we may need to use two or more intervals.
- Set up and evaluate the integral.

Example. Find the area of one of the regions between the graphs of $\sin(x)$ and $\cos(x)$.

Solution. Recalling that the sin and cos functions are periodic, we see that their are infinitely many regions between the graphs and each appears to have the same area. The graphs intersect when $\sin(x) = \cos(x)$ and using our knowledge of the values of sin and cos at the special values, we see that the first two points of intersection have $x = \pi/4$ where $\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$ and $x = 5\pi/4$ where $\sin(5\pi/4) = \cos(5\pi/4) = -\sqrt{2}/2$. On the interval $[\pi/4, 5\pi/4]$ we have $\cos(x) \leq \sin(x)$.

Thus, using (1) the area of one region is

$$\int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\, dx.$$

Evaluating the integral using the fundamental theorem we find that

$$\int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\, dx = -\cos(x) - \sin(x)\Big|_{x=\pi/4}^{5\pi/4} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}.$$

As an exercise, check the area of the next region and verify that it has the same area. ∎

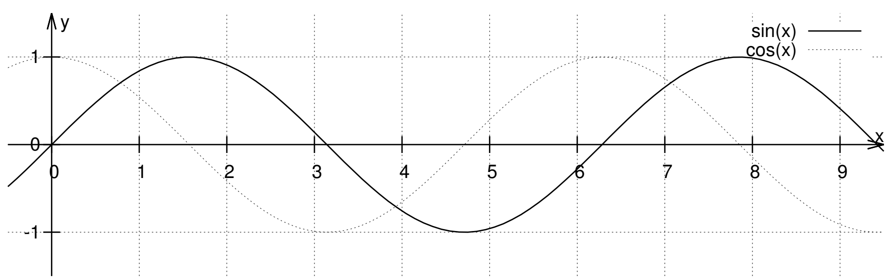

Figure 1: The graphs of $\sin(x)$ and $\cos(x)$

Example. Find the area between the curves $y = x + 2$ and $y = x^2$.

Solution. The curves intersect when $x+2=x^2$ or $x^2-x-2=0$. We factor to find the solutions of this equation, $x^2-x-2=(x-2)(x+1)=0$. Thus the solutions are $x=-1$ or $x=2$. On the interval $[-1,2]$, we have $x^2 \le x+2$. Thus, from (1) the area is

$$\int_{-1}^{2}(x+2-x^2)\,dx = \frac{x^2}{2}+2x-\frac{x^3}{3}\Big|_{x=-1}^{2} = (2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}) = 15/2-9/3 = 9/2.$$

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1.3 Using the y-axis

There are situations where it is simpler to compute the area by consider slices perpendicular to the y axis, rather than the x-axis. If we have the region $S=\{(x,y) : a \le y \le b,\ f(y) \le x \le g(y)\}$, then the area of S is given by

$$\text{Area of } S = \int_a^b g(y)-f(y)\,dy. \tag{2}$$

This approach is most useful when the boundary curves are given in the $x=f(y)$ or can be put into this form.

Example. Find the area between the line $y=x$ and $x=\sqrt{y}$.

Solution. The curves $y=x$ and $x=\sqrt{y}$ cross when $y=\sqrt{y}$ or $y=0$ or 1. From (2), the area is

$$\int_0^1 \sqrt{y}-y\,dy = (\frac{2}{3}y^{3/2}-\frac{1}{2}y^2)\Big|_{y=0}^{1} = \frac{2}{3}-\frac{1}{2} = 1/6.$$

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1.4 More examples

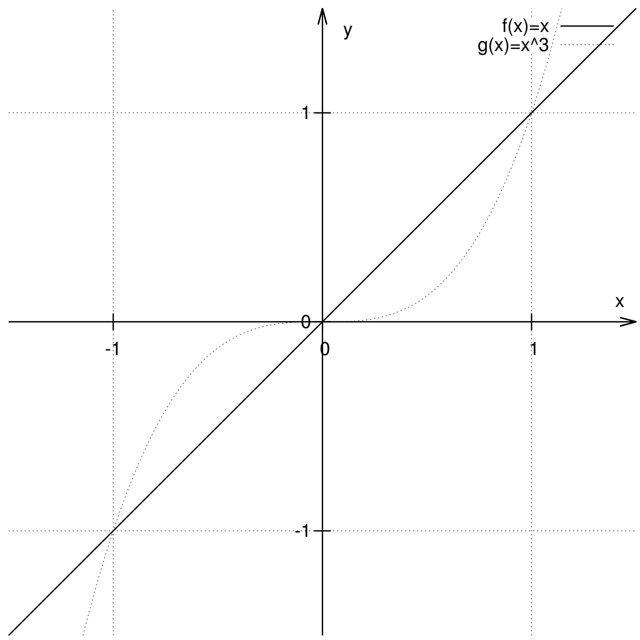
Example. Find the region between the graphs of $y=x$ and $y=x^3$.

Solution. From a sketch, we see that the graphs cross three times and solving $x^3=x$, the crossings occur for x in the set $\{-1,0,1\}$. We have $x<x^3$ for $-1<x<0$ and $x^3<x$ for $0<x<1$. Thus we can express the area between the curves as

$$\int_{-1}^{0} x^3-x\,dx + \int_0^1 x-x^3\,dx.$$

Evaluating the integrals using the first part of the Fundamental theorem gives

$$\int_{-1}^{0} x^3-x\,dx = \frac{1}{4}x^4-\frac{1}{2}x^2\Big|_{x=-1}^{0} = -\frac{1}{4}+\frac{1}{2} = \frac{1}{4}.$$



By a similar argument,

$$\int_0^1 x - x^3\,dx = \frac{1}{4}.$$

Thus altogether the area is $1/4 + 1/4 = 1/2$.

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