Problem Find the anti-derivative

$$\int \frac{1}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}\,dx$$

Solution. We assume that $a > 0$ and begin by factoring out $a\cos^2 x$ from the denominator.

$$\int \frac{1}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}\,dx = \frac{1}{a}\int \frac{1}{cos^2 x}\frac{1}{\tan^2 x + \frac{b}{a}\tan x + c}\,dx$$

If we substitute $u = \tan x$, $du = \frac{1}{\cos^2 x}dx$, we obtain

$$\frac{1}{a}\int \frac{1}{cos^2 x}\frac{1}{\tan^2 x + \frac{b}{a}\tan x + \frac{c}{a}}\,du = \frac{1}{a}\int \frac{1}{u^2 + \frac{b}{a}u + \frac{c}{a}}\,du$$

Now, we complete the square in the denominator and obtain

$$\frac{1}{a}\int \frac{1}{u^2 + \frac{b}{a}u + \frac{c}{a}}\,du = \frac{1}{a}\int \frac{1}{(u+\frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2}}\,du.$$

At this point, we consider several cases depending on the sign of $\frac{c}{a} - \frac{b^2}{4a^2}$ which has the same sign as $4ac - b^2$.

Case 1. If $b^2 - 4ac = 0$, then we may integrate by the power rule.

$$\frac{1}{a}\int \frac{1}{(u+\frac{b}{2a})^2}\,du = -\frac{1}{a}\frac{1}{u+\frac{b}{2a}} + C = -\frac{2}{(2a\tan x + b)} + C.$$

Case 2. If $b^2 - 4ac < 0$ In this case, we set $\alpha = \sqrt{\frac{c}{a} - \frac{b}{4a^2}}$. Then we may rewrite the integral and integrate using $\tan^{-1}$.

$$\begin{aligned}
\frac{1}{a}\int \frac{1}{(u+\frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2}}\,du &= \frac{1}{a}\int \frac{1}{(u+\frac{b}{2a})^2 + \alpha^2}\,du \\
&= \frac{1}{a\alpha}\tan^{-1}\left(2au + b\sqrt{4ac - b^2}\right) + C. \\
&= \frac{1}{a\alpha}\tan^{-1}\left(2a\tan x + b\sqrt{4ac - b^2}\right) + C.
\end{aligned}$$

Case 3. If $b^2 - 4ac > 0$. Then we set $\beta = \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$ and we may rewrite our integral as

$$\frac{1}{a}\int \frac{1}{(u+\frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2}}\,du = \frac{1}{a}\int \frac{1}{(u+\frac{b}{2a})^2 - \beta^2}\,du$$

We substitute $v = u + \frac{b}{2a}$, $du = dv$ to simplify this further which gives

$$\frac{1}{a}\int \frac{1}{(u+\frac{b}{2a})^2 - \beta^2}\,du = \frac{1}{a}\int \frac{1}{v^2 - \beta^2}\,dv.$$

We find the partial fractions decomposition of the integrand

$$\frac{1}{v^2-\beta^2} = \frac{1}{2\beta}\left(\frac{1}{v-\beta} - \frac{1}{v+\beta}\right).$$

And thus,

$$\begin{aligned}
\frac{1}{a}\int \frac{1}{v^2-\beta^2}\,dv &= \frac{1}{2\beta a}(\ln|v-\beta| - \ln|v+\beta|) + C \\
&= \frac{1}{2\beta a}\ln\left|\frac{v-\beta}{v+\beta}\right| + C \\
&= \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2a\tan x + b - \sqrt{b^2-4ac}}{2a\tan x + b + \sqrt{b^2-4ac}}\right| + C
\end{aligned}$$