1 Lecture: Applications of Taylor series

- Approximating functions by Taylor polynomials.
- Using Taylor's theorem with remainder to give the accuracy of an approximation.

1.1 Introduction

At several points in this course, we have considered the possibility of approximating a function by a simpler function. For example, the geometric problem which motivated the derivative is the problem of finding a tangent line. That is finding a linear function which is close to a function f at a. We made use of this approximation when derived Newton's method for solving an equation $f(x) = 0$. Solving $f(x) = 0$ is too hard. We replace f by its tangent line, $L(x)$ at given point and solve the easier equation $L(x) = 0$.

Those who study physics will run across the following calculation. If we let $\theta(t)$ denote the angular displacement of a pendulum at time t, then this displacement satisfies the differential equation

$$\theta''(t) = -k^2 \sin(\theta(t)).$$

Here k is a constant which depends on the pendulum. This equation is hard to solve. It is common to replace this equation by the simpler equation

$$\theta''(t) = -k^2\theta(t).$$

Why is this reasonable? The tangent line to $\sin(x)$ at $x = 0$ is the linear function x. It is plausible that the solutions of the simpler equation are close to the solutions of the original equation, at least if the displacements are small. Proving this is correct is a very interesting exercise–if only the semester were 40 weeks long!

One might also ask why the second equation is simpler. This is easy to see. The solutions of the second equation are of the form $A\sin(kx) + B\cos(kx)$ where A and B are constants.

1.2 Function approximation

We continue with a much simpler example, where we can give every detail. Suppose that we want to know the value of a function such as e^x. We can approximate the function e^x (which we can only evaluate on a calculator) and replace it by a polynomial which we can evaluate using pencil and paper.

Before going on, we introduce one new term. The *Taylor polynomial at a of degree n for a function f* is the terms in the Taylor series up to degree n, thus this polynomial, $T_n(x)$ is defined by

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Suppose we want to approximate e^x, or say e. We know from the previous section that e^x is given by the Taylor's series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. So that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

This series converges very rapidly and we shall see that we only need a few terms to find e to several decimal places of accuracy.

Example. Compute e to within an error of at most 10^{-3}.

Solution. According to Taylor's theorem with remainder, there is a number c between 0 and 1 so that

$$e - \sum_{n=0}^{N} \frac{1}{n!} \leq \frac{e^c}{(N+1)!}.$$

Take the largest and smallest values for e^c implies

$$\frac{1}{(N+1)!} e - \sum_{n=0}^{N} \frac{1}{n!} \leq \frac{e}{(N+1)!}. \tag{1}$$

Thus we need to find N so that $e/(N+1)! \leq 10^{-3}$. We apparently have a problem. In order find N and compute e, we need to know the value of e. There are several ways out of this circle. (1) Cheat–use the value of e from your calculator. This will be acceptable on tests. (2) Recall that when we defined e, we showed that $e \leq 4$. (3) In the example below, we will show how to use Taylor's theorem to find some information about the size of e. We temporarily use (2) and thus the error is at most $4/(N+1)!$.

Consider the table below:

n	$4/(N+1)!$	$\sum_0^N \frac{1}{n!}$
4	$4/120 = 0.0333$	
5	$4/720 = 5.5 \cdot 10^{-3}$	
6	$4/5040 = 7.9 \cdot 10^{-4}$	2.71805...

We see from this table that $N = 6$ is the smallest value that will give allow us to approximate e with an error of at most 10^{-3}. And, if I entered the numbers correctly in my calculator the approximate value is 2.71805.

∎

Example. Use Taylor's theorem to get a rough bound on e.

Solution. We use the displayed equation (1) from the previous example with $N = 2$ and that $e^c \leq e$ to obtain

$$\frac{1}{6} \leq e - \frac{5}{2} \leq e/6.$$

The left inequality gives

$$\frac{8}{3} \leq e.$$

While the right-hand one gives

$$\frac{5}{6}e \leq \frac{5}{2}$$

which implies $e \leq 3$. Thus we conclude

$$\frac{8}{3} \leq e \leq 3.$$

∎

We consider an example for the cosine function.

Example. Use Taylor's theorem to find an interval where

$$|\cos(x) - (1 - \frac{x^2}{2})| \leq 10^{-4}.$$

Solution. We recognize that $1 - x^2/2$ is the Taylor polynomial of degree 2 for cosine at 0, or the McLaurin polynomial for cos.

From Taylor's theorem, we have that

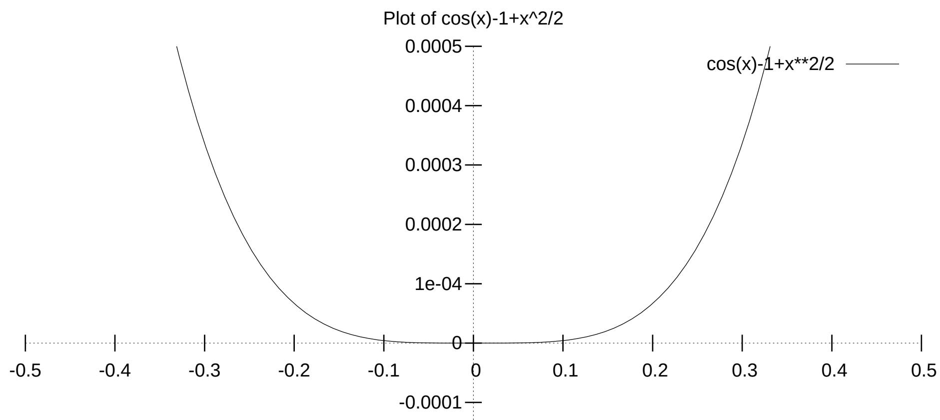
$$|\cos(x) - (1 - \frac{x^2}{2})| = |\sin(c)\frac{x^3}{3!}|.$$

Since we know that $|\sin(c)|$ is at most 1, the error will be at most 10^{-4} if we have that

$$|\frac{x^3}{3!}| \leq 10^{-4}.$$

Solving this inequality gives $|x| < 0.084...$

We can graph $\cos(x) - 1 + x^2/2$ and see if we have done a good job.


Plot of cos(x)-1+x^2/2
cos(x)-1+x**2/2
0.0005
0.0004
0.0003
0.0002
1e-04
0
-0.0001
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5

Examining this graph shows that we have not done a good job. The error does not become larger than 10^{-4} until x is about 0.2.

A moment's reflection will explain why we did not get the best possible answer. The second and third Taylor polynomials are equal because the terms of odd powers are 0. Thus, applying Taylor's theorem to estimate the difference between $\cos(x)$ and its Taylor polynomial of degree three gives us

$$|\cos(x) - (1 - x^2/2)| \leq x^4/4!.$$

Solving this inequality gives us

$$|x| \leq 0.22.$$

This agrees with the graph. ∎