1 Lecture: Properties of Logarithms

- Compute integrals and derivatives involving logarithms.
- Use definition to estimate $logx$ and establish properties.
- Compute integrals and derivatives involving exponential functions
- Find solutions of constant coefficient differential equations when the solution is of the form e^x.

Definition. If $x > 0$, we define the *natural logarithm of* x, $\ln x$ *by*

$$\ln x = \int_1^x \frac{dt}{t}.$$

Example. Show that $\frac{1}{2} \leq \ln x \leq 1$.

Solution. (Picture needed) By comparing the integral with the two rectangles pictured, we can see that

$$\frac{1}{2} \leq \ln 2 \leq 1.$$

∎

Question: How can you do better?

1.0.1 Properties of the logarithm

Proposition 1 *For* $x \neq 0$,

$$\frac{d}{dx} \ln |x| = 1/x.$$

Proof. For $x > 0$, this follows from FTC I.

For $x < 0$, use that $\frac{d}{dx}|x| = -1$ and the chain rule:

$$\frac{d}{dx} \ln |x| = \frac{-1}{|x|} = 1/x, \qquad \text{if } x < 0.$$

∎

Example. Compute$\int \frac{1}{t+1}\, dt$ and $\frac{d}{dx} \ln x^2 + 1$.

Proposition 2 *If $x > 0$ and $y > 0$, then we have*

- $\ln 1 = 0$
- $\ln xy = \ln x + \ln y$
- $\ln(\frac{x}{y}) = \ln x - \ln y$
- $\ln x^r = r \ln x$

Proof. $\ln 1 = \int_1^1 \frac{dx}{x}$

To establish the second, fix $y > 0$ and consider $\ln xy - \ln x - \ln y$. We compute that $g'(x) = 0$ and we know $g(1) = 0$ so $g(x) = 0$ for $x > 0$.

The last two are left for you to work out. ∎

Definition. The number e is defined as the unique solution of the equation

$$\ln e = 1$$

Note that e exists by the intermediate value theorem.

Since $\ln 2 \leq 1 \leq \ln 4$, and $\ln x$ is increasing, we have that $2 \leq e \leq 4$.

Proposition 3

$$\lim_{x \to \infty} \ln x = \infty$$

Proof. We have that

$$\ln 2^n = n \ln 2 \geq \frac{n}{2}.$$

Thus, $\lim_{n \to \infty} \ln 2^n = +\infty$. Since $\ln x$ is monotone increasing, we have that $\ln x$ also has the same limit at infinity. ∎

Exercise. What is the $\lim_{x \to 0^+} \ln x$?

Example. Sketch the graph of $\ln x$ and give the domain and range.

Solution. Graph is omitted. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$. ∎

Example. Compute the following.

$$\int \frac{\ln x}{x}\, dx \quad \int \cot x \, dx \quad \int \frac{x+1}{x^2 + 2x + 7}\, dx$$

1.0.2 Logarithmic differention

A good trick. Sometimes it is simpler to compute the derivative of $\ln y$ and then simplify to find y'. This technique is usually used on functions which are a product (or quotient) of simpler functions. The technique works because the logarithm converts multiplication into addition and thus avoids the product rule for differentiation.

Example. Find y' for $y = \frac{x}{x^2+1}$.

Solution.

$$\ln y = \ln x - \ln(x^2 + 1)$$

Hence, we have that

$$\frac{y'}{y} = \frac{1}{x} - \frac{2x}{x^2 + 1}.$$

Simplifying, gives that

$$y' = y(\frac{1}{x} - \frac{2x}{x^2 + 1}) = \frac{1}{x^2 + 1} - \frac{2x^2}{(x^2 + 1)^2}.$$

∎