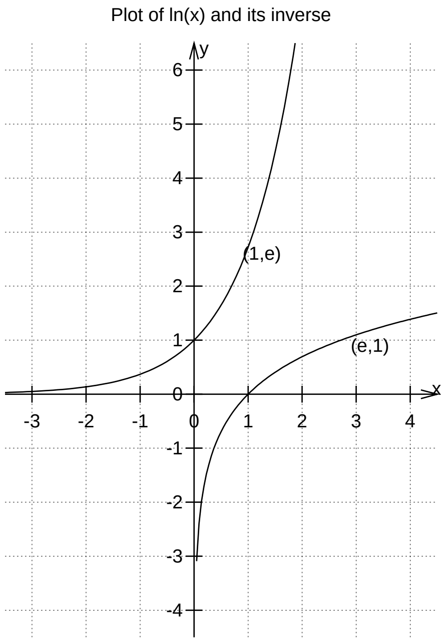
1 Lecture: Properties of exponentials

- Compute integrals and derivatives involving exponential functions
- Find solutions of constant coefficient differential equations when the solution is of the form e^x.

Recall the "cancellation rules": $f \circ f^{-1}(x) = x$ and $f^{-1} \circ f(x) = x$.

Definition. We define the exponential function e^x to be the inverse function of $\ln x$.



Example. What are the domain and range of e^x?

Solution. The domain of $\ln x$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. For e^x, these are reversed, the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$. ∎

Proposition 1

$$\frac{d}{dx} e^x = e^x \qquad \int e^x \, dx = e^x + C$$

Proof. First solution. If $g(x)$ is an inverse to $\log x$, then we have $\log g(x) = x$. Differentiating this expression, gives

$$g'(x) \frac{1}{g(x)} = 1$$

and then solving gives $g'(x) = g(x)$.

Second solution. According to the formula for the derivative of an inverse function, we have

$$f^{-1\prime}(x) = \frac{1}{f'(f^{-1}(x))}$$

Here, we have $f(x) = \log x$, $f'(x) = 1/x$ and $f^{-1}(x) = e^x$ so that

$$f^{-1\prime}(x) = e^x.$$

The formula for the integral follows from the formula for anti-derivatives.

∎

Example. Where are the inflection points for e^{-x^2}?

Find the integrals:

$$\int xe^{x^2}\,dx \qquad \int e^{x^2}\,dx.$$

Proposition 2 *(algebraic properties) If x and y are real numbers, then*

- $e^{x+y} = e^x e^y$
- $e^{x-y} = e^x/e^y$
- $(e^x)^y = e^{xy}$

Proof. Let $x = \log a$ and $y = \log b$. Then, $e^x e^y = e^{\log a} e^{\log b} = ab = e^{\log(ab)} = e^{\log a + \log b} = e^{x+y}$. We have used the properties of log and the cancellation conditoins.

Second and third are exercises. ∎

Remark. If one can follow through all the words, one should see that these properties are a restatement of the properties of logarithms. ∎

Example. Show that $y = \frac{e^x}{e^x+1}$ is one-one. Find the inverse function.

Solution. $y' = e^x/(1+e^x)^2$. Since $y' > 0$ on the interval $(-\infty, \infty)$, then y is increasing on this interval.

Solving $y = e^x/(1+e^x)$ gives that $x = \log y(1-y)$ for $0 < y < 1$. ∎

Example. For what values of r does e^{rx} solve

$$y'' + 2y' - 3y = 0, \qquad y'' + y = 0.$$