1 Exponential growth and decay.

- Set up and solve problems related to exponential growth and decay, including problems about half-life.
- Solve the differential equation $y' = ky$.

1.1 Examples of exponential growth or decay.

Example. Critters. Suppose that in a population of critters, 3% of the critters give birth each year and 2% of the critters die each year. Write an equation relating the population at time t, P and its derivative P'.

Solution. The rate of change of P with respect to time is given by $0.03P - 0.02P$ critters/year where the first term is the increase due to births and the second is the decrease due to deaths.

Since derivative is mathspeak for rate of change, we have

$$P' = 0.01P.$$

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Example. Apparently in a sample radioactive material, a fixed fraction of the material will spontaneously decay in each unit of time. Write a differential equation that describes the mass of material at time t.

Solution. If $M(t)$ is the mass at time t, then at time t, the rate of change of M is $-kM$ where k is a positive constant. Thus, we have

$$\frac{dM}{dt} = -kM.$$

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Remark: Note that (most of) the mass does not disappear. Rather, it is converted to another isotope or element. Thus, M in this equation refers to the mass of one particular isotope.

Example. Suppose a population of critters doubles every hour. If a glass of critters is full at 12 noon on 15 September 2001, when is it half full?

1.2 A useful differential equation.

One solution of the equation $y' = ky$ is obvious: $y(t) = e^{kt}$. Is this the only one?

Theorem 1 *If $y' = ky$ in an open interval, then $y(t) = Ae^{kt}$ on that interval.*

Proof. Consider $g(t) = y(t)/e^{kt}$. Compute $g'(t) = 0$. Hence g is a constant on any interval where it is defined. Call the constant A. ∎

Note that if $k > 0$ e^{kt} grows and if $k < 0$, the e^{kt} decays.

Example. Show that all solutions of $y' = -2xy$ are of the form Ae^{-x^2}.

1.3 A problem.

Example. If a population grows exponentially, and the population is 100 at 1pm, while it is 144 at 3pm, find the population at 4pm.

Solution. Let $P(t)$ denote the number of critters at t hours after noon.

We have $P' = kP$ and hence $P(t) = Ae^{kt}$. We need to find A and k. This seems like a plausible task since we have two pieces of information that we can use to determine the two constants.

We have the equations $Ae^k = 100$ and $Ae^{3k} = 144$. Together these give $e^{2k} = 1.44$ and hence $e^k = 1.2$. Since $Ae^k = 100$, $A = \frac{5}{6}100 \approx 83.3$. Hence $P(t) = \frac{5}{6}100e^{kt}$ and $P(4) = \frac{5}{6}1001.2^4 = 172.8$.

Second solution:

Since $e^{2k} = 1.44$, $2k = \ln 1.44$ and thus $k \approx 0.1823$. Next, we have $Ae^k = 100$, so $A = 100/e^k \approx 100/e^{.1823} \approx 83.335$. Then, $P(t) = Ae^{kt}$ with A and k as above.

Finally, $P(4) \approx 83.335 * e^{.1823 \cdot 4} \approx 172.8$.

Note in both cases, we should probably round to 173 since you can't have 0.8 of a critter.

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1.4 Half-life

The *half-life* of a substance is the amount of time it takes for a radio-active substance to decrease to one-half of its original mass. Thus $M(t+T) = \frac{1}{2}M(t)$ for T the half-life.

Because of the properties of the exponential function, we have

$$Ae^{-(t+T)} = \frac{1}{2}Ae^{-t}$$

is equivalent to $e^{-kT} = \frac{1}{2}$ or

$$kT = \ln 2.$$

DON'T MEMORIZE THIS.

Example. Suppose Kryptonite 123 (Kr_{123}) has a third-life of 1023 years. How long will it take for 100 grams of Kryptonite to decay to 25 grams?

Solution. We have $M(t) = Ae^{kt}$. Thus, $e^{-kT} = \frac{1}{3}$ and solving gives $k1023 = \ln 3$ or

$$k = \frac{\ln 3}{1023}.$$

Since we begin with 100 grams, we have $M(t) = Ae^{-kt}$. We want to solve $M(t_0) = 25$ or

$$e^{-kt_0} = \frac{1}{4}.$$

Solving gives $kt_0 = \ln 4$ or $t_0 = \frac{1}{k} \ln 4 = \frac{\ln 4}{\ln 3} 1023 \approx 1291$years. ∎