Lecture 7: Inverse trigonometric functions

- Use definitions of the inverse trigonometric functions, sin^{-1}, cos^{-1} and tan^{-1}.
- Simplify expressions of the form $f(g^{-1}(x))$ where f and g are trigonometric functions.
- Compute derivatives and integrals involving inverse trigonometric functions. One should know both the derivatives and understand how to use the material from section 6.1 to find these derivatives.

0.1 Definitions of the inverse trigonometric functions.

The functions $\sin x$, $\cos x$, $tanx$ and $\sec x$ are periodic and cannot be one to one. In order to define the inverse functions, we must restrict the domains. In order to prevent chaos, there are conventions about the way to do this–at least for sin, cos and tan. These conventions are:

$\sin x$	$[-\pi/2, \pi/2]$
$\cos x$	$[0, \pi]$
$\tan x$	$(-\pi/2, \pi/2)$
$\sec x$	$[0, \pi/2) \cup [\pi, 3\pi/2)$

Each of these functions is one-to-one on the given domain, and thus we can define an inverse function. You can check that the inverse function keys on your calculator correspond to these definitions of $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$. The $\sec^{-1}$ function is not easily available on your calculator and in fact one can find different definitions in different calculus textbooks.

Example. If $\cos(u) = 4/5$, find $\sin(u)$ and $\tan(u)$. Why is this impossible? Can you answer the question if we know $\pi < u < 2\pi$?

Solution. To find $\sin u$, we recall that $\sin^2 x + \cos^2 x = 1$. Solving this identity for sin gives

$$\cos u = \pm\sqrt{1 - \sin^2 u}.$$

Thus, we cannot know $\cos u$ unless we have some information that helps us choose the sign.

If we know that $\pi < u < 2\pi$, then we can see that $\cos u < 0$ so we should choose the negative sign.

Thus, $\cos u = -\sqrt{1 - (4/5)^2} = -3/5$.

Finding the tangent is easy:

$$\tan u = \sin u / \cos u = -4/3.$$

∎

Exercise. Can you simplify $\cos \sin^{-1} x$?

Solution. Yes, $\sqrt{1-x^2}$. $\blacksquare$

Example. Is it always true that $\sin^{-1} \sin x = x$?

Solution. No. What if $x = 2\pi$? $\blacksquare$

0.2 Derivatives of the inverse trigonometric functions.

We begin with $\sin^{-1}$ and draw a sketch of function $\sin x$ and its inverse.

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Observe that both functions are increasing.

Next, we list the domain and range of these functions.

Example. Compute the derivative of $\sin^{-1} x$.

Solution. For convenience, we let $g(x) = \sin^{-1} x$. As in section 6.1, we start with $\sin)g(x) = x$ for x in $[-1, 1]$. We differentiate both sides and obtain

$$g'(x)\cos(g(x)) = 1.$$

Solving and using that $\cos g(x) = \sqrt{1-x^2}$ gives the important formula:

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}. \tag{1}$$

Note that when we choose the positive square root because $\cos(g(x)) \geq 0$ if $-\pi/2 \leq x \leq \pi/2$. $\blacksquare$

Now we do the same for $\tan^{-1}$.

Example. Find the derivative of $\tan^{-1}(x)$.

Solution. Again, we set $g(x) = \tan^{-1} x$ and differentiate

$$\tan(g(x)) = x.$$

We differentiate and solve for $g'(x)$ which gives

$$g'(x) = \frac{1}{\sec^2(g(x))}.$$

If we recall the identity $\sec^2(g(x)) = 1 + \tan^2(g(x)) = 1 + x^@$, we can simplify this to obtain

$$g'(x) = \frac{1}{1+x^2}.$$

$\blacksquare$

0.3 Some integrals

With these differentiation formulae we can now evaluate a few new integrals:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} x/a + C$$

and

$$\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1} x/a + C.$$

Example. Check these.

Solution. One may differentiate or substitute. ∎

Example. Compute that following integrals.

1. $\int \frac{x}{\sqrt{4-x^2}}\, dx$
2. $\int \frac{1}{x^2+x+5}\, dx$
3. $\int \frac{1}{\sqrt{4-x^2}}\, dx$

0.4 A geometric problem.

The trigonometric functions relate the sides of a right triangle to angles in the triangle. One place where these functions arise in nature are when we try to relate angular to linear velocity.

Example. Suppose that a disco ball hangs in the center of a square room whose walls are 40 meters long. A beam of light reflected off one of the mirrors moves along the wall. Suppose that a reflect light beam moves along the wall at 120meters/minute when it is 25 meters from the disco ball. Find the angular velocity of the disco ball.

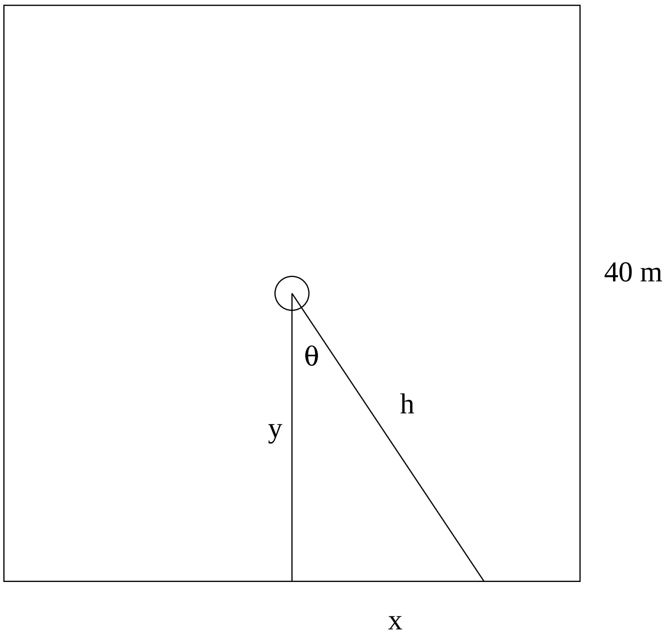
Solution. We let x y h and θ be as in the picture. We know that that the length y is fixed and write $x/y = \tan\theta$ or $\theta(t) = \tan^{-1}(x/y)$.

Differentiating gives

$$\theta'(t) = \frac{x'}{y} \frac{1}{1 + \frac{x}{y}^2}.$$

From Pythagoras's theorem, $x = 15$meters and we are given that $x' = 120$meters/minute and $y = 20$. Thus, we have

$$\theta'(t) = \frac{120}{20} \frac{1}{1 + (3/4)^2} = 6\frac{16}{25} \text{ minute}^{-1} = \frac{96}{25} \text{ minute}^{-1}$$



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Remark. Notice that in this problem, one can compute the units by writing units for each entry in θ' cancelling in the same way that we simplify algebraic expressions. This example illustrates that radians are a dimensionless quantity, a radian is the quotient of two lengths and thus it is not necessary to include it in the final answer. ∎

0.5 Exercises

1. Describe how you would use your calculator to find $\sec^{-1}(2)$ and $\sec^{-1}(-3)$.