Lecture 8: L'Hopital's rule

- Recognize indeterminate forms. Especially, $0/0$, ∞/∞. Be able to reduce limits of the form $0 \cdot \infty$, 1^0 and 1^∞ to evaluating a limit of a quotient.
- State L'Hopital's rule for quotients.
- Compute limits using L'Hopital's rule.

Some well-known limits

Recall some familiar limits:

$$\begin{aligned} \lim_{x \to 0} \frac{\sin(2x)}{x} &= 2 \\ \lim_{x \to 0} \frac{\ln(1+x)}{x} &= 1 \\ \lim_{x \to 0} \frac{1-\cos(x)}{x^2} &= \frac{1}{2} \end{aligned}$$

The first two may be viewed as difference quotients and this allows us to know the limit. For example,

$$\frac{\sin(2x)}{x} = \frac{\sin(2x) - 0}{x - 0}$$

and thus with $f(x) = \sin(2x)$,

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = f'(0) = 2$$

Indeterminate forms

Each of the limits above can be thought of as

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0.$$

This is an *indeterminate form of type* $\frac{0}{0}$.

As the above examples show, these limits can have many values depending on the functions f and g. This is why the expression $f(x)/g(x)$ is called an indeterminate form.

Exercise. Given a value a, can you choose f and g so that the indeterminate form has the value a?

If in the above limit we have

$$\lim_{x\to a} f(x) = \infty \quad \text{and} \quad \lim_{x\to a} g(x) = \infty$$

then this is an indeterminate form of type $\frac{\infty}{\infty}$. (We say the same if one or both of the limits is $-\infty$.

Example.

$$\lim_{x\to\infty} \frac{e^x}{x}.$$

Indeterminate forms of type 0∞. If

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} g(x) = \infty$$

then

$$\lim_{x\to a} f(x)g(x)$$

is an indeterminate form. This can be rewritten as $0/0$ by considering $f(x)/(g(x)^{-1})$.

Exercise. Can you also rewrite this indeterminate form as $\infty/infty$?

Indeterminate forms of 1^∞. We rewrite

$$f(x)^{g(x)} = e^{g(x)\ln(f(x))}.$$

If f approaches 1, then $\ln f$ approaches 0 and we assume that g has a limit of ∞ Thus on the right-hand side, the exponent is the indeterminate form $\infty \cdot 0$.

Example. Compute

$$\lim_{x\to 0}(1+x)^{1/x}.$$

Solution. We rewrite this as

$$\lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)}$$

since we evaluated the limit of the exponent above and the exponential function is continuous, we have

$$\lim_{x\to 0}(1+x)^{1/x} = e.$$

∎

The rule, finally.

Theorem 1 *If* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *is an indeterminate form of type* $0/0$ *or of type* ∞/∞*, and we have*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$$

then $\lim_{x\to a} \frac{f(x)}{g(x)}$ *exists and equals* L*.*

We will not give the proof here. If a is finite, then the rule can be proven by a generalization of the mean value theorem.

Example. Try to apply L'Hopital's rule to

$$\lim_{x\to 0} \frac{x}{e^x}.$$

What do we obtain? What is the value of the limit?

Solution. Misapplying L'Hopital's rule gives 1. L'Hopital's rule is not applcable because x/e^x is not an indeterminate form at 0. The correct value of the limit is 0. ∎

Examples

Example. Use L'Hopital to compute

$$\lim_{x\to 0} \ln(1+x)x, \quad \lim_{x\to\infty} \frac{e^x}{x}, \quad \lim_{x\to\infty} xe^{-x}, \quad \lim_{x\to 0} \frac{1-\cos x}{x^2}$$

Example. Show that

$$\lim_{x\to\infty} e^x x^n = \infty$$

for $n = 0, 1, 2, 3, \ldots$.

Solution. The base case $n = 0$ is obvious.

If we know that $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$, then by L'Hopital,

$$\lim_{x\to\infty} \frac{e^x}{x^{n+1}} = \lim_{x\to\infty} \frac{e^x}{(n+1)x^n}$$

If the limit is infinity for n, it will also be $+\infty$ for $n+1$.

Now the principle of mathematical induction tells us that the limit is $+\infty$ for all whole numbers n.

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