Lecture 10: Powers of $\sin$ and $\cos$

- Integrating non-negative powers of sin and cos.

The goal.

In this section, we learn how to evaluate integrals of the form

$$\int \sin^n x \cos^m x \, dx.$$

The procedure will depend on several familiar trigonometric identities.
and the double angle formula for $\cos x$,

Case 1. One of m or n is odd.

Let us suppose that $m = 2k + 1$ is odd. In this case, we rely on the Pythagorean identity,

$$\sin^2 x + \cos^2 x = 1$$

which allows to rewrite the integral

$$\int \sin^{2k+1} x \cos^n x \, dx.$$

as

$$\int \sin x(1 - \cos^2 x)^k \cos^n x \, dx.$$

This is sort of a mess, but you should be able to see that the substitution $u = \cos x$ will reduce this to evaluating the integral of a polynomial.

Exercise. How does the procedure differ in n is odd and m is even?

This will be clear if we try an example.

Example. Find the indefinite integral

$$\int \sin^3 x \cos^3 x \, dx.$$

Solution. Since both exponents are odd, there are at least two ways to evaluate the integral. We choose to rewrite $\cos^2 x$ as $1 - \sin^2 x$ and then substitute $u = \sin x$, $du = \cos x\, dx$. This gives

$$\begin{aligned}
\int \sin^3 x \cos^3 x\, dx &= \int \sin^3 x (1 - \sin^2 x) \cos x\, dx \\
&= \int u^3 (1 - u^2)\, du \\
&= \int u^3 - u^5\, du \\
&= \frac{u^4}{4} - \frac{u^6}{6}\, du \\
&= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C\, du
\end{aligned}$$

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Case 2. Both exponents are even.

For this case, we rely on the double-angle formula for cos.

$$\begin{aligned}
\cos(2x) &= 2\cos^2 x - 1 && (1) \\
&= 1 - 2\sin^2 x. && (2)
\end{aligned}$$

In fact, we will take these two forms and solve them for $\cos^2 x$ and $\sin^2 x$, respectively. This gives us the formulae

$$\begin{aligned}
\cos^2 x &= \frac{1 + \cos 2x}{2} && (3) \\
\sin^2 x &= \frac{1 - \cos 2x}{2} && (4) \\
& && (5)
\end{aligned}$$

Many of you will recognize these as the half-angle formula.

To make use of these formulae, we will start with an integral of the form

$$\int \sin^{2k} x \cos^{2j} x\, dx$$

If we substitute the formulae (3) (4), we end up with

$$\int (\frac{1 - \cos 2x}{2})^k (frac1 + \cos 2x2)^j\, dx.$$

If we multiply out the powers, we obtain an expression involving $\sin 2x$ and $\cos 2x$ which is of LOWER DEGREE. If we repeatedly use this trick, we end up with terms that we can treat by the method of case 1.

Again, an example.

Example. Evaluate
$$\int_0^{2\pi} \sin^2 x\, dx.$$

Solution. We first find an anti-derivative. Using the double-angle formula (4) we have
$$\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx.$$
This integral is comparatively easy to evaluate
$$\int \frac{1-\cos 2x}{2}\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$
Some of you may wish to use the substitution $u = 2x$. Now we make use of the anti-derivative
$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$
to evaluate the definite integral.
$$\int_0^{2\pi} \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4} \Big|_{x=0}^{2\pi} = \pi.$$
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Exercise. Compute the integrals
$$\int \sin^2 x \cos^2 x\, dx \quad \int \sin^5 x\, dx \quad \int \tan x \sec^2 x\, dx \quad \int \tan x\, dx.$$

We close with an example which illustrates where such integrals arise in nature.

Example. The voltage of an alternating electric current might have the form
$$V(t) = A\sin(\omega t).$$
Because the voltage changes over time, it is not clear how to assign a single number that represents the voltage of this current. For example, the 110 volt current in our houses is actually an alternating current where the voltage may reach as high as 155 volts.

A standard way of assigning a single voltage to V is to take the root-mean-square voltage. This means, square V, take mean or average over one period and then take the square root. This gives
$$RMS = \left(\frac{\omega}{2\pi} \int_0^{2\pi/\omega} V(t)^2\, dt \right)^{1/2}.$$

a) Compute RMS. b) Show that if A is 155, then RMS is about 110.

Solution. A short calculation shows

$$RMS = A/\sqrt{2}$$

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An even shorter calculation shows $155/\sqrt{2}$ is about 109.6. ∎