1 Lecture: Rationalizing substitutions

- Substitution $u = \sqrt[n]{ax+b}$.
- The Weierstraß substitution, $u = \tan(x/2)$. (Not to be examined.)

In the previous section, we described an algorithm that will let us integrate any rational function. In this section, we learn a few substitutions that will allow us to convert integrals that we do not yet know how to do into rational functions.

2 The substitution $u = \sqrt[n]{ax+b}$.

If $R(v)$ is a rational function, then we may find the anti-derivative

$$\int R(\sqrt[n]{ax+b})\,dx$$

by the substitution $u = \sqrt[n]{ax+b}$.

We consider an example:

Example. Evaluate

$$\int \frac{1}{\sqrt{x}+1}\,dx.$$

Solution. We let $u = \sqrt{x}$. If we compute $du\frac{1}{2\sqrt{x}}\,dx$, we might be confused as to what do next since dx is not multiplied by $\frac{1}{2\sqrt{x}}$ in the integral.

There are several ways around this problem. One is to solve $u = \sqrt{x}$ for x giving $x = u^2$. Then we can write $dx = 2udu$. (This may seem like something new, but in fact we used a similar technique when we substituted $x = \sin u$ a few days ago.)

With these substitutions, we have

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{2u}{1+u}\,du.$$

The integral on the right is the integral of a rational function. To evaluate the integral, we divide to write the integrand as a sum of a polynomial and a proper rational function. This gives:

$$\int \frac{2u}{1+u}\,du = \int 2 + \frac{2}{1+u}\,du.$$

We can evaluate the anti-derivative on the right, to obtain.

$$\int 2 + \frac{2}{1+u}\,du = 2u + \ln|1+u| + C.$$

Replacing u by $\sqrt{x}$ gives

$$\int \frac{1}{1+\sqrt{x}}\,dx = 2\sqrt{x} + \ln|1+\sqrt{x}| + C.$$

∎

2.1 The Weierstraß substitution.

It is possible to evaluate any rational expression in $\cos x$ and $\sin x$. In this section, we explain how to do this.

The key to this method is an ingenious substitution that allows to express both $\sin x$ and $\cos x$ as rational functions.

We begin by setting $u = \tan(x/2)$. Recalling that $\cos^2(x/2) = 1/sec^2(x/2) = 1/(1+\tan^2(x/2))$, we obtain that

$$\cos^2(x/2) = \frac{1}{1+u^2}.$$

If we use the double angle formulae, then

$$\cos(x) = (2\cos^2(x/2) - 1) = \frac{2}{1+u^2} - 1 = \frac{1-u^2}{1+u^2}.$$

It is perhaps a bit of a surprise that

$$\sin x = \sqrt{1-\cos^2(x)} = \frac{2u}{1+u^2}.$$

And then, we have $x = 2\tan^{-1} u$ so that

$$dx = \frac{1}{1+u^2} du.$$

Using this substitution, it is clear that any rational expression in sin and cos becomes a rational function in u.

Example. Find the antiderivative.

$$\int \frac{1}{2+\cos x}\, dx$$

Solution. With the substitution

$$\sin x = \frac{2u}{1+u^2}, \quad dx = \frac{2}{1+u^2}\, du$$

we obtain that

$$\int \frac{1}{2+\cos x}\, dx = \int \frac{1}{2+\frac{2u}{1+u^2}} \frac{1}{1+u^2}\, du = \int \frac{1}{2+2u^2+2u}\, du$$

This integral, we can evaluate:

$$\frac{1}{2}\int \frac{1}{1+u+u^2}\, du = \frac{1}{2}\int \frac{1}{(u+\frac{1}{2})^2+\frac{3}{4}}\, du = \frac{2}{\sqrt{3}} \tan^{-1}(\frac{2}{\sqrt{3}}(u+\frac{1}{2}) + C.$$

Thus, in the end, we obtain:

$$\int \frac{1}{2+\cos x}\, dx = \frac{2}{\sqrt{3}} \tan^{-1}(\frac{2}{\sqrt{3}}(\tan(x/2)+\frac{1}{2}) + C.$$

∎

3,4,5

The Weierstraß substitution has another interesting application that I should not talk about because this is a calculus class and the application is in algebra. However, let us live dangerously.

The relations

$$\cos x = \frac{1-u^2}{1+u^2} \qquad \sin x = \frac{2u}{1+u^2}$$

imply that for any value of u, the point

$$\left(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}\right)$$

lies on the unit circle. If we substitute a rational number for u, then we end up with a point on the unit circle which has rational coordinates. If we clear the denominators, this gives us an integer solution of $a^2 + b^2 = c^2$.

To work out the details, we let $u = m/n$ and if we simplify to clear all fractions in

$$\left(\frac{1-(m/n)^2}{1+(m/n)^2}\right)^2 + \left(\frac{2(m/n)}{1+(m/n)^2}\right)^2 = 1$$

we obtain

$$(n^2 - m^2)^2 + (2mn)^2 = (m^2 + n^2)^2.$$

Substituting $m = 1$, $n = 2$ gives the familiar relation

$$3^2 + 4^2 = 5^2.$$

And $m = 2$ and $n = 3$ gives

$$5^2 + 12^2 = 13^2.$$

Exercise. Does the expression $(m^2 - n^2, 2mn, m^2 + n^2)$ give all integer solutions of the equation $a^2 + b^2 = c^2$?

Exercise. Can you find an integer solutions of $a^3 + b^3 = c^3$?, $a^4 + b^4 = c^4$?