1 Lecture 15: Strategy

Today, we will review the techniques of integration we have learned and consider a few examples that can be treated by each technique.

1.1 The table

Know the entries 1-17 on the table and integrals that are a simple substitution away.

1.2 Integration by parts.

There are three basic types of integrals we may treat by integration by parts.

1. A polynomial multiplied by a trigonometric or exponential functions.
2. A product of sin or cos and an exponential function.
3. Logarithm or inverse trigonometric functions.

In addition, the technique of integration by parts is important in proving reduction formulae.

Example. Several integrals that can be treated are:

$$\int x \sin x \, dx \quad \int e^x \sin x \, dx \quad \int \tan^{-1} x \, dx.$$

1.3 Trigonometric integrals

Here there are two basic approaches.

1. To integrate $\int \sin^n x \cos^m x \, dx$ when n is odd, one may substitute $u = \sin x$ and make use of the Pythagorean identity.
2. To integrate $\int \cos^n x \sin^m x \, dx$ when both exponents are even, we use the double-angle formula to lower the power.

Example.

$$\int \sin^2 x \, dx \int \frac{\sin x}{\cos x} \, dx$$

1.4 Trigonometric substitution

Substitute $u = a \sin x$ to evaluate integrals involving $\sqrt{a^2 - x^2}$.

Example.

$$\int \sqrt{2x - x^2} \, dx \int \frac{1}{(4 - 9x^2)^{3/2}} \, dx \int x\sqrt{4 - x^2} \, dx.$$

1.5 Rational functions

Use the method of partial fractions.

1.6 Rationalizing substitutions

Substitute $u = \sqrt[n]{ax+b}$ for integrals involving this expression.

Example.

$$\int \frac{1}{\cos x} \qquad \int \frac{1}{1+\sqrt[3]{x}}\, dx.$$

Solution. For the first integral, we observe that we have $\cos x$ raised to an odd, but negative power. Still, we can proceed as follows:

$$\int \frac{1}{\cos x}\, dx = \int \frac{1}{\cos x}\, dx = \int \frac{\cos x}{1-\sin^2 x}\, dx.$$

Next, we substitute $u = \sin x$ which gives

$$\int \frac{1}{\cos x}\, dx = \int \frac{1}{1-u^2}\, du.$$

We make the easy partial fractions decomposition which gives

$$\frac{1}{1-u^2} = \frac{1}{2}(\frac{1}{1+u} + \frac{1}{1-u}).$$

Thus,

$$\int \frac{1}{1-u^2}\, du = \frac{1}{2} \ln|1+u| - \ln|1-u| + C.$$

And finally, we have

$$\int \frac{1}{\cos x}\, dx = \frac{1}{2} \ln\left|\frac{1+\sin x}{1-\sin x}\right|\, dx.$$

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Exercise. Show that the answer we obtained above is equivalent to the standard expression for the integral of $\sec x$.