1 Lecture 18: Separable differential equations

1. Solve initial value problems for separable differential equations.

2. Solve mixing problems where the volume is contant.

Exponential growth and more

In this section, we learn how to solve differential equations of the form

$$\frac{dy}{dt} = f(y)g(t). \tag{1}$$

Such equations are *separable differential equations*. To solve such an equation means to find a function $y(t)$ which makes the above equation true. We find our solutions by integrating which introduces a constant of integration. (Now you know why $+C$ is so important.) Typically, we can choose the constant to fix the value of $y(0)$ as we did when we studied exponential growth and decay.

The pair of equations

$$y' = f(y)g(t) \quad y(0) = y_0$$

is called an *initial value problem.*

Separating variables

To solve the equation 1, we use the technique of separating variables. This means we move all the y's to one side and all the t's to the other. If we then integrate, we obtain the equation

$$\int \frac{dy}{f(y)} = \int g(t)\, dt.$$

We work a simple example.

Example. Solve

$$y' = 1 - y, \quad y(0) = 2.$$

Find $\lim_{t\to\infty} y(t)$.

Solution. $y(t) = 1 + e^{-x}$.

Note that the limiting value $y(t) = 1$ is a solution of $y' = 1 - y$. ∎

We give a more interesting example.

Example. Solve

$$\frac{dy}{dt} = y(10 - y) \quad y(0) = 5.$$

Find $\lim_{t\to\infty} y(t)$.

Solution. We separate variables, find the partial fractions decomposition of the expression involving Y and integrate to obtain

$$\int \frac{dy}{y(10-y)} = \int dt$$

and then

$$\frac{1}{10}\int \frac{1}{y} + \frac{1}{10-y} = t + C.$$

which implies

$$\ln\left|\frac{y}{10-y}\right| = t + C$$

Using that $y(0) = 5$ implies

$$\frac{y}{10-y} = e^t.$$

$$y(t) = \frac{10e^t}{1+e^t}.$$

Thus,

$$\lim_{t\to\infty} y(t) = 10.$$

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Talk about direction field and why this is not so hard to see.

Mixing problems

Example. Suppose a tank contains 100 liters of water and brine with a concentration of 3 grams of salt per liter is flowing in at a rate of 2 liters /minute. The tank is well-stirred and brine flows out at a rate of 2 liters/minute. Find the mass of salt in the tank at time t.

Solution. We have the differential equation for $M(t)$, the mass of salt in the tank at time t,

$$\frac{dM}{dt} = 6 - \frac{M}{50}.$$

where 6 grams/minute is the salt flowing in and $2M/100$ is the salt flowing out.

Solving this differential equation with the initial value $y(0) = 0$ gives

$$M(t) = 300(1 - e^{-t/50}).$$

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