1 Lecture 19: Arc length–graphs

1. Compute lengths of curves which are presented as graphs.

A definition of arc-length

By now, we know how to find the length of a line segment. If the coordinates of the endpoints are (x_1, y_1) and (x_2, y_x), then the theorem of Pythagoras tells us that we can find the length of the line segment joining these points by the formula

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We will use this simple fact to give a definition of the length of a curve. To do this, we begin by approximating the curve by an inscribed polygonal path. If the curve is of the form $\{(x, f(x)) : a \leq x \leq b\}$, then we partition the interval $[a, b]$ with $P = \{a = x_0 < x_1 < x_2 \ldots < x_n = b\}$. We let $\Delta x_i = x_i - x_{i-1}$ and then we connect the points$\{(x_i, f(x_i)) : i = 0 \ldots n\}$ to form a polygonal curve whose length is

$$S = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (f(x_i) - f(x_{i-1}))^2} \approx \sum_{i=1}^{n} \sqrt{1 + f'(t_i)^2} \Delta x_i.$$

This looks suspiciously like a Riemann sum for an integral and if we let the mesh of the partition tend to zero, we obtain the *arc length* of this curve is

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Some examples

We begin by showing that this definition agrees with what we learned in school for lines and circles.

Example. Find the length of the line $y = 2x$ for $1 \leq x \leq 4$.

Example. Find the length of the semi-circle $y = \sqrt{9 - x^2}$, $0 \leq x \leq 3$.

Example. Find the length of the curve $y = x^{3/2}$ for $0 \leq x \leq 1$.

Solution.

$$\int_0^1 \sqrt{1 + 9x/4}\, dx. = \frac{8}{27}((13/4)^{3/2} - 1) \approx 1.44$$

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