UNIQUENESS IN THE INVERSE CONDUCTIVITY PROBLEM FOR NONSMOOTH CONDUCTIVITIES IN TWO DIMENSIONS

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Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with Lipschitz boundary and let $\gamma : \Omega \to \mathbf{R}$ be a function which is measurable and bounded away from zero and infinity. We consider the divergence form elliptic operator

$$L_\gamma = \mathrm{div}\gamma\nabla.$$

It is well-known that for such a operator, we may solve the Dirichlet problem

$$\begin{cases} L_\gamma u = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases}$$

for all f in the Sobolev space $H^{1/2}(\partial\Omega)$. The resulting solution lies in the Sobolev space $H^1(\Omega)$. Using this solution, we define the Dirichlet to Neumann map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}.$$

Here, $\frac{\partial u}{\partial \nu}$ is the normal derivative of u at the boundary. In general, the expression $\gamma\frac{\partial u}{\partial \nu}$ exists as an element of $H^{-1/2}(\partial\Omega)$ defined by

$$\langle \gamma \frac{\partial u}{\partial \nu}, \Psi \rangle = \int_\Omega \gamma \nabla u \cdot \nabla \Psi$$

when Ψ is in $H^1(\Omega)$.

In this work, we establish the following uniqueness result (see Corollary 5.3): If γ_1 and γ_2 are two conductivities with $\nabla\gamma_i$ in $L^p(\Omega)$, $p > 2$, and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$. The main interest of this result is the weakened regularity hypothesis on the γ_i. Uniqueness in the inverse conductivity problem for smooth conductivities was established in 1987 by G. Uhlmann and J. Sylvester [18]. This result was extended to conductivities with two derivatives by a number of authors, see [10, 12], for example. S. Chanillo has studied a related problem for Schrödinger equations [6] with nonsmooth potential. In addition, Nachman's work [10] and work of R.G. Novikov [13] give a method for reconstructing the conductivity (or the potential in an equivalent problem for Schrödinger operators) using $\bar\partial$-techniques and integral equations. Recently, R. Brown showed that Sylvester and Uhlmann's methods could be extended to conductivities which are $C^{3/2+\epsilon}$ [5]. C. Tolmasky studies more general problems using methods of pseudo-differential operators with nonsmooth symbols [19]. All of these results are in dimensions 3 and higher. In two dimensions, A. Nachman [11] was able to prove uniqueness for conductivities with two derivatives using the $\bar\partial$-method. The hypotheses in the above results (and the result of this paper) imply, via Sobolev embedding, that the conductivity is continuous. It is interesting to note that the only uniqueness results available for conductivities which are discontinuous are due to Kohn and Vogelius [9] who study conductivities which are piecewise analytic and V. Isakov [8] who considers a class of conductivities which are piecewise C^2.

The proof given here parallels Nachman's two-dimensional result in that we use the $\bar\partial$-method. However, rather than studying a second order Schrödinger equation, we follow work of Beals and Coifman [2, 3] and others who study scattering for a first order system

$$\left[\begin{pmatrix} \bar\partial & 0 \\ 0 & \partial \end{pmatrix} - \begin{pmatrix} 0 & q \\ \bar q & 0 \end{pmatrix}\right]\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = 0.$$

The new ingredient that allows us to establish uniqueness under less regularity is a sharper version of Liouville's theorem for pseudo-analytic functions (see

Corollary 3.11) and the observation of Beals and Coifman [3] that the scattering data (see part iv) of Theorem A below) lies in $L^2(\mathbf{R}^2)$. Using these ingredients and a technical estimate for the scattering solutions due to L. Sung [15], we prove uniqueness for the special solutions which easily gives the uniqueness result for the inverse problem. We note that for smoother conductivities, Beals and Coifman's results lead, via the arguments presented below, to uniqueness in the two-dimensional inverse conductivity problem without this sharper version of the Liouville theorem.

1 A first order system.

In this section, we introduce a first order system which is related to the conductivity equation $L_\gamma u = 0$. The scattering theory for this first-order system has been developed in Beals and Coifman [3]. We recall the notation and ideas of this paper. A more detailed analysis of the scattering theory for this system was given by L. Sung [15, 16, 17] with the intention of studying the Davey-Stewartson II system. Sung and Beals and Coifman study slightly different, but equivalent, formulations of this system. We will use Beals and Coifman's formulation and notation.

For this section, we suppose that we have an elliptic operator $\text{div}\gamma\nabla$ where the coefficient γ satisfies $\nabla\gamma \in L^2_{loc}(\mathbf{R}^2)$ and for some $\delta > 0$ we have $\delta < \gamma < \delta^{-1}$ a.e. in $\mathbf{R}^2$. Additional hypotheses will be imposed in later sections. Note that we assume that γ is defined in all of $\mathbf{R}^2$. In section 4 we will explain how we pass from γ defined in Ω to γ defined in all of $\mathbf{R}^2$. We recall the standard notation for complex derivatives:

$$\bar\partial = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}) \text{ and } \partial = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}).$$

We let $q = -\frac{1}{2}\partial \log\gamma$ and define a matrix potential Q by

$$Q = \begin{pmatrix} 0 & q \\ \bar q & 0 \end{pmatrix}.$$

We let D be the operator

$$D = \begin{pmatrix} \bar\partial & 0 \\ 0 & \partial \end{pmatrix}$$

and observe that if u satisfies the equation

$$\text{div}\gamma\nabla u = 0,$$

then a calculation shows that

$$\begin{pmatrix} v \\ w \end{pmatrix} = \gamma^{1/2} \begin{pmatrix} \partial u \\ \bar{\partial} u \end{pmatrix}$$

satisfies the system

$$D \begin{pmatrix} v \\ w \end{pmatrix} - Q \begin{pmatrix} v \\ w \end{pmatrix} = 0.$$

We let

$$J = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

and for a 2×2 matrix A, we let

$$A^{\mathit{off}} = \begin{pmatrix} 0 & a^{12} \\ a^{21} & 0 \end{pmatrix} \text{ and } A^{d} = \begin{pmatrix} a^{11} & 0 \\ 0 & a^{22} \end{pmatrix}$$

be the off-diagonal and diagonal parts of the matrix. We define

$$\begin{aligned} \mathcal{J} A &= [J, A] \\ &= 2JA^{\mathit{off}} = -2A^{\mathit{off}} J \end{aligned}$$

where $[\ ,\ ]$ denotes the commutator. For z and k in $\mathbf{R}^2$ (though we use complex numbers to denote points in $\mathbf{R}^2$), we let

$$\begin{aligned} \Lambda(z,k) &= \Lambda_k(z) \\ &= \begin{pmatrix} \exp(iz\bar{k} + i\bar{z}k) & 0 \\ 0 & \exp(-izk - i\bar{z}\bar{k}) \end{pmatrix} \end{aligned}$$

and then we define a map on 2×2 matrix valued functions $A = A(z)$ by

$$\begin{aligned} E_k A &= A^d + \Lambda_k^{-1} A^{\mathit{off}} \\ &= A^d + A^{\mathit{off}} \Lambda_{\bar{k}}. \end{aligned}$$

We observe that the free system $D\psi = 0$ has a family of solutions

$$\begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}$$

depending on the complex parameter k. As in [3], we look for special solutions of the system

$$(D - Q)\psi = 0 \tag{1.1}$$

of the form

$$m(z,k)\begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix} \tag{1.2}$$

where m is a matrix valued function of z and k and $m(z,k)$ goes to 1 as $z \to \infty$ in a sense to be made precise. Here and below, we use 1 to denote the 2×2 identity matrix. We observe that ψ satisfies the equation (1.1) if and only if m satisfies

$$D_k m - Qm = 0 \tag{1.3}$$

where D_k is the operator

$$\begin{aligned} D_k A &= E_k^{-1} D E_k A \\ &= DA + k\mathcal{J}A. \end{aligned}$$

We fix the inverse of D as

$$D^{-1}f(z) = \frac{1}{\pi}\int_{\mathbf{R}^2} \begin{pmatrix} z-\zeta & 0 \\ 0 & \bar{z}-\bar{\zeta} \end{pmatrix}^{-1} f(\zeta)\, d\mu(\zeta)$$

where $d\mu$ is Lebesgue measure on $\mathbf{R}^2$ and then we look for solutions of (1.3) by studying the integral equation

$$m - D_k^{-1}Qm = 1 \tag{1.4}$$

where $D_k^{-1} = E_k^{-1}DE_k$.

2 Construction of solutions and the $\bar{\partial}$ equation.

In this section, we review the scattering theory for the system (1.1). Most of these facts are well-known, though perhaps not quite in the form stated here. Thus, we state the results we need and give a sketch of the proof and references to earlier work. The main ideas here are in work of Beals and Coifman [3]. We follow their argument and add a few analytical details borrowed from Nachman [11]. A detailed analysis along the same lines has been given by L. Sung for potentials in $L^1 \cap L^\infty$ and much of his argument extends to potentials in L^p_c. Here and below, we use L^p_c to denote the subspace of L^p which consists of functions which are compactly supported. We say that $Q_j \to Q$ in L^p_c

if $\|Q_j - Q\|_{L^p}$ goes to zero and the sequence Q_j is supported in a fixed ball, independent of j. We will also use the space L^p_α which is the space of functions f for which the norm $\|f\|_{L^p_\alpha} = \|(1+|\cdot|^2)^{\alpha/2} f\|_{L^p}$ is finite.

Throughout this paper, we assume that the potential Q satisfies $Q^d = 0$. Additional hypotheses will be stated explicitly in each result. Typically, we will require that Q be Hermitian and that $Q \in L^p_c$. Our first result gives the existence of solutions to (1.4), the $\bar\partial$-equation satisfied by these solutions and a few miscellaneous facts which will be useful.

Theorem A *Suppose that $Q \in L^p_c$ for some $p > 2$ and that $Q^* = Q$. Choose r so that $1/r + 1/p > 1/2$ and then β so that $\beta r > 2$.*

i) Then we may construct m as

$$(I - D_k^{-1}Q)^{-1}(1)$$

where the inverse is taken on the space $L^r_{-\beta}$.

ii) The map $k \to m(\cdot, k)$ is continuous into $L^r_{-\beta}$ and for each k fixed, the map $Q \to m(\cdot, k)$ is continuous from L^p_c into $L^r_{-\beta}$.

iii) For each k, the equation (1.3) has a unique solution subject to the condition that $m(\cdot, k) - 1$ is in L^q for some q, $2 < q < \infty$.

iv) The map $k \to m(\cdot, k)$ is differentiable (as a map into $L^r_{-\beta}$) and the derivative satisfies

$$\frac{\partial}{\partial \bar k} m(z,k) - m(z,\bar k)\Lambda_k(z) S(k) = 0.$$

where the scattering data S is defined by

$$S(k) = -\frac{1}{\pi}\mathcal{J}\int_{\mathbf{R}^2} E_k(Q(z)m(z,k))\, d\mu(z). \tag{2.1}$$

Sketch of Proof. We first note that our choice of r and β guarantees that $L^\infty \subset L^r_{-\beta}$. Next, we note that we have $\|D_k^{-1} f\|_{L^\infty} \le C(q)(\|f\|_{L^q} + \|f\|_{L^{q'}})$ for any $q \ne 2$. Using this, our condition on r and the assumption that Q has compact support gives that

$$\|D_k^{-1}Qf\|_{L^r_{-\beta}} \le C\|Q\|_{L^p}\|f\|_{L^r_{-\beta}} \tag{2.2}$$

where the constant depends on p, r, $-\beta$ and the diameter of the support of Q. Furthermore, one can show that the map $f \to D_k^{-1}Qf$ is compact (see [11, Lemma 4.2] for a similar result). If $m(z) \in L^r_{-\beta}$ satisfies the integral equation $m - D_k^{-1}Qm = 0$, then it is clear that in fact m lies in all L^p spaces for $p > 2$ and C_0. Thus one may use the Liouville theorem for pseudo-analytic functions as in [3, p. 22] to show that $m = 0$. (See also Corollary 3.8 below or Sung's work [15].) Since $I - D_k^{-1}Q$ has trivial kernel, the Fredholm theory implies that for each k, the operator $I - D_k^{-1}Q$ is invertible. This argument also gives the uniqueness statement in part iii). The continuity of the map $Q \to m$ in part ii) follows from the observation (2.2) that $Q \to D_k^{-1}Q$ is continuous into the operators on $L^r_{-\beta}$ in the operator norm. The continuity of the map $k \to m$ follows from the differentiability asserted in iv).

To establish the differentiability, we may argue as in [11, Lemma 2.2] to show that $k \to D_k^{-1}Qf$ is a differentiable function taking values in $L^r_{-\beta}$ when f is in $L^r_{-\beta}$. Computing the derivative of this map by differentiating under the integral and then using the quotient rule gives that

$$\begin{aligned}\frac{\partial}{\partial \bar{k}} m(z,k) &= -\frac{1}{\pi}(I - D_k^{-1}Q)^{-1}\left(\Lambda_k(z)\mathcal{J}\int_{\mathbf{R}^2} E_k Q (I - D_k^{-1}Q)^{-1}(1)\, d\mu\right) \\ &= (I - D_k^{-1}Q)^{-1}(\Lambda_k(z)S(k)).\end{aligned}$$

Now as in [3], one may compute that $D_k(AE_k^{-1}S) = (D_{\bar{k}}A)E_k^{-1}S$ and then that $(I - D_k^{-1}Q)^{-1}(\Lambda_k S) = ((I - D_{\bar{k}}^{-1}Q)(1))\Lambda_k S(k)$, which gives the equation of part iv). ∎

Theorem B *If $Q \in L^p_c$, for some $p > 2$, and $Q = Q^*$, then*

$$\int \operatorname{tr} SS^* \leq \int \operatorname{tr} QQ^*.$$

We also note that in [16, Corollary 4.20], Sung gives the conclusion of Theorem B above with an equality instead of an inequality when the potential in $L^1 \cap L^\infty$. The proof of this result depends on the ideas used in Sung's Proposition 2.23 of [15], as corrected below.

Proof. For Q with entries in the Schwartz class, it is known [3, 16] that

$$\int \operatorname{tr} SS^* = \int \operatorname{tr} QQ^*$$

where tr denotes the trace of a matrix. We observe that from the observation ii) in Theorem A and the definition of S it is clear that if $Q_j \to Q$ in L^p_c, then the scattering data for Q_j converges pointwise to the scattering data for Q. Thus, if we approximate a general potential by a sequence of smooth, compactly supported potentials, then Fatou's lemma and the equality above give that the scattering data for a general potential in L^p_c satisfies the estimate of this theorem. ∎

Note that Theorem A does not give any estimate for the growth of m, beyond the observation that $\|m(\cdot,k)\|_{L^r_{-\beta}}$ is locally bounded in k. Our next theorem gives such an estimate. At the same time, we show that $m(z,\cdot)$ is in L^q for some finite q. This technical result will be important in our uniqueness proof for the inverse problem. The estimate of Theorem 2.3 is a straightforward generalization of Sung's [15, Proposition 2.23]. However, there is an error in the proof of Sung's Proposition 2.23. L. Sung was kind enough to provide us with a correction. The main step of his correction, Lemma C, is presented below. This lemma, which is also valid for potentials in $L^1 \cap L^\infty$, may be used to solve the integral equation (2.52) in [15]. We make a similar use of this lemma in proving Theorem 2.3 below. We thank Sung for allowing us to use his correction and for several useful communications.

Theorem 2.3 *Suppose $Q^* = Q$ and $Q \in L^p_c$ for some $p > 2$, then for all $q > 2p/(p-2)$,*

$$\sup_z \|m(z,\cdot) - 1\|_{L^q} \leq C$$

where the constant C depends on p, q and Q.

The main step in the proof of this theorem, is the following lemma. In this lemma and below, we use $L^p_z(L^q_k)$ to denote mixed L^p spaces.

Lemma C [L. Sung] *Given $Q \in L^p_c$, for some $p > 2$, there is an $R = R(Q)$ so that the map*

$$m \to D^{-1} Q D_k^{-1} Q m$$

is a contraction on the diagonal matrix valued functions in $L^\infty_z(L^q_k(\{k : |k| > R\}))$, $1 \leq q \leq \infty$.

Proof of Theorem 2.3. Note that if we set $\ell = m^d - 1$, then ℓ satisfies the integral equation

$$\ell - D^{-1} Q D_k^{-1} Q \ell = D^{-1} Q D_k^{-1} Q.$$

In deriving this equation, it is helpful to note that the operator $D_k^{-1}Q$ simplifies to $D^{-1}Q$ when acting on off-diagonal matrices. We claim that the right-hand side of this integral equation lies in $L_z^\infty(L_k^q)$, $q > 2p/(p-2)$. Accepting the claim, we may then use Lemma C to obtain that $m^d - 1$ lies in $L_z^\infty(L_k^q(\{k : |k| > R\}))$ for some R. Note that this conclusion does not depend on the hypothesis $Q^* = Q$.

From Theorem A, we have that $\|m(\cdot,k)\|_{L^r_{-\beta}}$ is continuous and hence locally bounded. Then the hypothesis that $Q \in L_c^p$ and the integral equation (1.4) imply that $\|m(\cdot,k)\|_{L^\infty}$ is locally bounded. This implies that $m^d - 1 \in L_z^\infty(L_k^q)$.

Now we turn to the proof of the claim. Here, we write out one entry in the matrix $m_2 = D^{-1}QD_k^{-1}Q$. The other entry may be handled in the same way. Explicitly, we have

$$\begin{aligned} &m_2^{11}(z,k) \\ &\quad = \int \frac{Q^{12}(z_1)}{(z-z_1)} \int \frac{Q^{21}(z_2)}{(\bar z_1 - \bar z_2)} \exp(-ik(z_1-z_2) - i\bar k(\bar z_1 - \bar z_2))\, d\mu(z_2) d\mu(z_1). \end{aligned}$$

To see that this is in $L_z^\infty(L_k^q)$, we use the integral form of Minkowski's inequality to bring the L_k^q-norm inside the first integral and then the Hausdorff-Young inequality to see that the inner integral gives a function in L_k^q, $q > 2p/(p-2)$. This follows because, if $Q \in L_c^p$, then the expression $Q^{21}(\cdot)/(\bar z_1 - \bar\cdot)$ lies in $L^{q'}$ for q' in the dual range.

Finally, we show that m^{off} is in $L_z^\infty(L_k^q)$. To do this, we write $m^{\mathit{off}} = D_k^{-1}Q(1) + D_k^{-1}Q(m^d - 1)$. The first term is in $L_z^\infty(L_k^q)$ by the Hausdorff-Young inequality and the second term may be shown to lie in $L^\infty(L_k^q)$ using Minkowski's integral inequality and the above observation that $m^d - 1 \in L_z^\infty(L_k^q)$. ∎

Proof of Lemma C. To estimate the operator norm, we consider only one component. The other component is treated in the same manner. We write

$$Tg(z) = \int A(z, z_2; k) Q^{21}(z_2) g(z_2, k)\, d\mu(z_2)$$

where now g is a scalar valued function and A is defined by

$$A(z, z_2; k) = \int \frac{Q^{12}(z_1)}{(z-z_1)(\bar z_1 - \bar z_2)} \exp(-ik(z_1 - z_2) - i\bar k(\bar z_1 - \bar z_2))\, d\mu(z_1).$$

From Minkowski's integral inequality, we have

$$\begin{aligned}
&\|Tg(z,\cdot)\|_{L^q(\{k:|k|>R\})}\\
&\quad\leq \int \sup_{\{k:|k|>R\}} |A(z,z_2;k)Q(z_2)|\, d\mu(z_2) \sup_{z_2} \|g(z_2,\cdot)\|_{L^q(\{k:|k|>R\})}.
\end{aligned}$$

Thus, we turn to a study of the integral in the previous expression which dominates the operator norm. First, we note that a direct calculation gives that

$$\int_{|z-z_2|<\delta} \sup_k |A(z,z_2;k)||Q^{21}(z_2)|\, d\mu(z_2) \leq C\delta^{1-2/p}\|Q\|^2_{L^p} \tag{2.4}$$

where C depends on p and the size of the support of Q. Next we note that part ii) of Sung's Lemma A.1 [15] or a classical work of Vekua [20] gives that

$$\lim_{|z|\to\infty} \int \sup_k |A(z,z_2;k)||Q^{21}(z_2)|\, d\mu(z_2) = 0. \tag{2.5}$$

Since Q is compactly supported, we have for R sufficiently large that

$$\int_{|z_2|>R} \sup_k |A(z,z_2;k)||Q^{21}(z_2)|\, d\mu(z_2) = 0. \tag{2.6}$$

Note that each of the three previous assertions did not use the oscillatory exponential in the definition of A. This cancellation is used in our last observation. Fix $R_0 > 0$ and $\delta > 0$, then we have

$$\lim_{R\to\infty} \sup_{\substack{|k|>R,|z|<R_0,\\ |z_2|<R_0,|z-z_2|>\delta}} |A(z,z_2;k)| = 0. \tag{2.7}$$

This follows from the Riemann-Lebesgue Lemma and the observation that

$$\{\frac{Q^{12}(z_1)}{(z-z_1)(\bar{z}_1-\bar{z}_2)} : |z-z_2|>\delta, |z|<R, |z_2|<R\}$$

is a pre-compact set in L^1. Combining the observations (2.4), (2.5), (2.6) and (2.7) we see that we may choose R so that

$$\sup_z \int \sup_{|k|>R} |A(z,z_2;k)||Q^{12}(z_2)|\, d\mu(z_2) \leq 1/2$$

and thus $1/2$ dominates the operator norm of T on $L^\infty_z(L^q(\{k : |k|>R\}))$. ∎

3 Liouville's theorem for pseudo-analytic functions.

The main result of this section is Corollary 3.11 where we establish uniqueness for the pseudo-analytic equation $\bar\partial u = f\bar u$. Such a result is well-known when f is in $L^p \cap L^{p'}$, $p \neq 2$. See [3, 11, 15, 20]. The authors thank A. Nachman for telling them of the extension to $p = 2$, the proof below is due to the authors.

In the proof below, we need to consider $\bar\partial^{-1} f$ where f lies in $L^2(\mathbf{R}^2)$. This is best defined as an element of $BMO(\mathbf{R}^2)$ and is thus an equivalence class of functions which differ by constants. We arbitrarily fix a representative by setting

$$\bar\partial^{-1} f(z) = \frac{1}{\pi}\int_{\mathbf{R}^2} (\frac{1}{z-\zeta} - \frac{1}{\zeta}) f(\zeta)\, d\mu(\zeta).$$

Theorem 3.1 *Suppose f is in $L^2(\mathbf{R}^2)$ and $w \in L^p(\mathbf{R}^2)$, for some finite p and that $w\exp(-\bar\partial^{-1} f)$ is holomorphic. Then w is zero.*

Proof. In the proof below, we will let $u = -\bar\partial^{-1} f$. We observe that since f is in L^2, the gradient of u, ∇u, is in L^2 and we have

$$\|\nabla u\|_{L^2} = 2\|f\|_{L^2}. \tag{3.2}$$

We will use the notation $B_r(z)$ to denote the disk $\{\zeta : |z-\zeta| < r\}$ and for a disk B, we let

$$v_B = \mu(B)^{-1}\int_B v\, d\mu$$

be the average of a function v on B.

We begin by claiming that if we fix r, then

$$u_{B_r(z)} = o(\log|z|), \qquad \text{as } z \to \infty. \tag{3.3}$$

This is an immediate consequence of the following stronger result. There is an absolute constant C so that if $z, w \in \mathbf{R}^2$ and $r < s$, then

$$|u_{B_r(z)} - u_{B_s(w)}| \le C\|f\|_{L^2} \left(\log(|z-w|/s + s/r + 2)\right)^{1/2}. \tag{3.4}$$

We remark that we do not know if the square root in this result can be exploited. To prove (3.4), we first observe that if we have two balls, B_1 and B_2

with $\bar{B}_1 \cap \bar{B}_2 \neq \emptyset$ and $1/4 \leq \mu(B_1)/\mu(B_2) \leq 4$, then Poincaré's inequality gives that

$$|u_{B_1} - u_{B_2}| \leq C \left(\int_{\tilde{B}} |\nabla u|^2 \right)^{1/2} \tag{3.5}$$

where $\tilde{B}$ is the smallest disk containing $B_1 \cup B_2$. Now suppose that $B_r(z)$ and $B_s(w)$ are two arbitary disks with $r < s$. Then we may find a sequence of disks $B_0 = B_r(z), B_1, \ldots, B_N = B_s(w)$ with i) $\bar{B}_j \cap \bar{B}_{j-1} \neq \emptyset$, $j = 1, \ldots, N$, ii) $1/4 \leq \mu(B_j)/\mu(B_{j-1}) \leq 4$, $j = 1, \ldots, N$, iii) if $\tilde{B}_j$ is the smallest disk containing $B_j \cup B_{j-1}$, $j = 1, \ldots, N$, then $\sum_{j=1}^{N} \chi_{\tilde{B}_j} \leq C$ and iv) $N \leq C \log(|z - w|/s + s/r + 2)$. In conditions iii) and iv), C is an absolute constant and χ_E is the characteristic function of the set E. Now we apply (3.5) on each pair B_{j-1}, B_j and then use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} |u_{B_0} - u_{B_N}| &\leq C \sum_{j=1}^{N} \left(\int_{\tilde{B}_j} |\nabla u|^2 \, d\mu \right)^{1/2} \\ &\leq C\sqrt{N} \left(\int_{\cup \tilde{B}_j} |\nabla u|^2 \sum_{j=1}^{N} \chi_{\tilde{B}_j} \, d\mu \right)^{1/2}. \end{aligned}$$

Now (3.4) follows from iii), iv) and (3.2).

Next, we observe that since $\nabla u \in L^2$, it follows that

$$\lim_{r \to 0^+} \sup_{z \in \mathbf{R}^2} \int_{B_r(z)} |\nabla u|^2 \, d\mu = 0.$$

Thus, we may use Theorem 7.21 of [7] to conclude that, given $p > 1$, there exists $C > 0$ and $r_0 = r_0(p, f)$ so that if $r < r_0$, then

$$\int_{B_r(z)} \exp(p'|u - u_{B_r(z)}|) \, d\mu \leq C\mu(B_r(z)). \tag{3.6}$$

(An alternative proof can be given by observing that u has small BMO-norm on small disks and then appealing to the John-Nirenberg inequality, see [14, p. 154], for example.) Using Hölder's inequality, we have that for all disks B,

$$|(e^u w)_B| \leq |e^{u_B}| \mu(B)^{-1} \left(\int_B \exp(p'|u - u_B|) \, d\mu \right)^{1/p'} \left(\int_B |w|^p \, d\mu \right)^{1/p}.$$

We use that w is in L^p, (3.3) and (3.6) to conclude that there is an $r_0 = r_0(p, u)$ so that

$$|(e^u w)_{B_{r_0}(z)}| = o(|z|), \qquad \text{as } z \to \infty.$$

Since $e^u w$ is a holomorpic function, we may use the ordinary Liouville's theorem to conclude that $e^u w$ is constant or that $w = C_0 e^{-u}$.

Now we wish to show that $C_0 = 0$ and thus $w = 0$. To do this, we claim that

$$\left| \int_{B_r(0)} w \, d\mu \right| \leq \mu(B_r(0))^{1-1/p} \|w\|_p, \qquad r > 0 \tag{3.7}$$

and that for each $\epsilon > 0$, there exists an $R_0 > 0$ and $C > 0$ so that

$$\int_{B_r(0)} \exp(-u) \, d\mu \geq \mu(B_r(0))^{1-\epsilon} \exp(-C\|f\|_{L^2}), \qquad r > R_0. \tag{3.8}$$

Together, these two claims force that $C_0 = 0$. The inequality (3.7) is a straightforward consequence of Hölder's inequality. To establish (3.8), we observe that

$$\begin{aligned} \int_{B_r(0)} \exp(-u) \, d\mu &\geq \int_{B_r(0)} \exp(-|u|) \, d\mu \\ &\geq \mu(B_r(0)) \exp(-|u|_{B_r(0)}) \end{aligned} \tag{3.9}$$

where the second inequality is Jensen's inequality. The triangle inequality gives that

$$|u|_{B_r(0)} \leq |u - u_{B_r(0)}|_{B_r(0)} + |u_{B_r(0)}|.$$

Now Poincaré's inequality, (3.4) and (3.2) imply that given $\epsilon > 0$, there exists R_0 so that if $r > R_0$, then

$$|u - u_{B_r(0)}|_{B_r(0)} + |u_{B_r(0)}| \leq C\|f\|_{L^2} + \epsilon \log(\mu(B_r(0))).$$

Thus we have

$$\exp(-|u|_{B_r(0)}) \geq \exp(-C\|f\|_{L^2}) \mu(B_r(0))^{-\epsilon}, \qquad r > R_0. \tag{3.10}$$

Now (3.9) and (3.10) gives (3.8). ∎

Finally, a well-known argument gives the uniqueness for pseudo-analytic functions.

Corollary 3.11 *Suppose $u \in L^p(\mathbf{R}^2) \cap L^2_{loc}(\mathbf{R}^2)$ for some p, $1 \leq p < \infty$ and satisfies the equation*

$$\bar{\partial} u = au + b\bar{u}$$

where a and b lie in $L^2(\mathbf{R}^2)$. Then $u = 0$.

Proof. We define a function f by

$$f(z) = \begin{cases} a(z) + b(z)\bar{u}(z)/u(z), & u(z) \neq 0 \\ \\ 0, & u(z) = 0. \end{cases}$$

Then $\bar{\partial} u = fu$ and thus u is zero by the previous theorem. ∎

4 Relation between Λ_γ and S.

In this section, we show that if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then we may extend γ_1 and γ_2 to $\mathbf{R}^2 \backslash \bar{\Omega}$ in such a way that the scattering data S_1 and S_2 agree. Here, we are constructing the first order systems $D - Q_i$ from the conductivities γ_i as in section 1. In order to carry out this step, we impose the requirement that Ω be a Lipschitz domain. By this we mean that the boundary is locally the graph of a Lipschitz function.

Theorem 4.1 *Suppose Ω is a Lipschitz domain and that L_{γ_1} and L_{γ_2} are two divergence form operators for which $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. If $\gamma_1, \gamma_2 \in W^{1,p}(\Omega)$, $p > 2$, then we may extend γ_1 and γ_2 to $\mathbf{R}^2 \backslash \bar{\Omega}$ so that $\nabla \gamma_i$ are in L^p_c and so that the scattering data for the associated first order systems satisfies $S_1 = S_2$.*

Proof. Since $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ and $\gamma_1, \gamma_2 \in W^{1,p}(\Omega)$, we have that $\gamma_1 = \gamma_2$ on $\partial\Omega$ (see [1, 4]). Thus we may extend γ_1 and γ_2 to $\mathbf{R}^2 \backslash \bar{\Omega}$ so that $\gamma_1 = \gamma_2$ in $\mathbf{R}^2 \backslash \bar{\Omega}$, $\gamma_1 = \gamma_2 = 1$ outside a large ball, and $\nabla\gamma_1, \nabla\gamma_2 \in L^p(\mathbf{R}^2)$. This extension uses that $\partial\Omega$ is Lipschitz. Now that γ_j is defined in all of $\mathbf{R}^2$, we may construct a first order system $D - Q_j$ as in section 1. We let ψ_j be the special solutions for this system as in (1.2). Then for each j the scattering data $S_j(k)$ has the representation

$$\begin{aligned} S_j(k) &= -2J \int_{\mathbf{R}^2} \begin{pmatrix} 0 & \bar{\partial}\psi_j^{12} e^{-iz\bar{k}} \\ \partial\psi_j^{21} e^{i\bar{z}k} & 0 \end{pmatrix} d\mu(z) \\ &= -2J \left[\int_{\mathbf{R}^2 \backslash \Omega} \begin{pmatrix} 0 & \bar{\partial}\psi_j^{12} e^{-iz\bar{k}} \\ \partial\psi_j^{21} e^{i\bar{z}\bar{k}} & 0 \end{pmatrix} d\mu(z) \right. \\ &\quad \left. + \int_{\partial\Omega} \begin{pmatrix} 0 & \bar{\nu}\psi_j^{12} e^{-iz\bar{k}} \\ \nu\psi_j^{21} e^{i\bar{z}\bar{k}} & 0 \end{pmatrix} d\mu(z) \right] \end{aligned}$$

where we use $\nu = \frac{1}{2}(\nu_1 + i\nu_2)$ and $\bar{\nu} = \frac{1}{2}(\nu_1 - i\nu_2)$ with (ν_1, ν_2) the unit outer normal to $\partial\Omega$. From the second expression for S_j, we see that if we can show

$$\psi_1(z,k) = \psi_2(z,k) \quad \text{in } \mathbf{R}^2 \backslash \bar{\Omega}, \tag{4.2}$$

then $S_1 = S_2$.

We proceed to show (4.2) by a well-known argument (see [18, Lemma 2.1]). We claim that if γ_1 and γ_2 are as above and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\psi_1 = \psi_2$. To see this, we select a column of ψ_1 and call it φ. We let φ^1 and φ^2 be the columns of φ and set

$$v = \gamma_1^{-1/2}\varphi^1 \text{ and } \tilde{v} = \gamma_1^{-1/2}\varphi^2.$$

Since $(D - Q_1)\varphi = 0$, we have the compatibility condition

$$\bar{\partial} v = \partial \tilde{v}$$

and thus there exists a potential u satisfying $\partial u = v$ and $\bar{\partial} u = \tilde{v}$. Since $(D - Q_1)\varphi = 0$, we also have $L_{\gamma_1} u = 0$. Now, we set

$$\tilde{u} = \begin{cases} u, & \text{in } \mathbf{R}^2 \backslash \bar{\Omega} \\ \\ \tilde{u}_2, & \text{in } \Omega \end{cases}$$

where $\tilde{u}_2$ is the solution of the Dirichlet problem

$$\begin{cases} L_{\gamma_2} \tilde{u}_2 = 0 & \text{in } \Omega \\ \\ \tilde{u}_2 = u, & \text{on } \partial\Omega. \end{cases}$$

Since $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ and $\gamma_1 = \gamma_2$ in $\mathbf{R}^2 \setminus \bar{\Omega}$, we have $L_{\gamma_2}\tilde{u} = 0$ in $\mathbf{R}^2$. Thus if we set

$$\tilde{\varphi} = \gamma_2^{1/2} \begin{pmatrix} \partial \tilde{u} \\ \bar{\partial} \tilde{u} \end{pmatrix}$$

we have $(D - Q_2)\tilde{\varphi} = 0$ in $\mathbf{R}^2$. Since $\tilde{\varphi}$ and φ are equal outside Ω, it follows from the uniqueness assertion in Theorem A that $\tilde{\varphi}$ is a column of ψ_2. Note that initially, we only have that $\tilde{\varphi}$ is in L^2_{loc}. However, because $\tilde{\varphi}$ satisfies $(D - Q)\tilde{\varphi} = 0$ and Q is L^p, $p > 2$, we may apply standard local elliptic regularity results to conclude φ is actually continuous. Thus we have established (4.2) and hence the theorem. ∎

5 The uniqueness theorem.

In this section, we give the remaining details needed to prove our uniqueness theorem. The main result is Corollary 5.3. Theorem 5.1 shows that if the scattering data agree, then the special solutions agree. Theorem 5.2 gives one way of recovering the potential from these solutions.

Theorem 5.1 *Suppose that Q_j are two potentials satisfying $Q_j^* = Q_j$ and $Q_j \in L^p_c$, $p > 2$. Let S_j be the scattering data for Q_j. Then if $S_1 = S_2$, we have that $m_1 = m_2$.*

Proof. Let $S = S_1 = S_2$ be the common scattering data. According to Theorem A, we have $S \in L^2(\mathbf{R}^2)$. Now we recall the $\bar{\partial}$-equation for m_j in Theorem A. Since $S_1 = S_2$, we may subtract these equations and obtain that $m = m_1 - m_2$ satisfies

$$\frac{\partial}{\partial \bar{k}} m(z,k) = m(z,\bar{k})\Lambda_k(z)S(k).$$

As in [3], if we set

$$\begin{aligned} u_\pm(k) &= m^{11}(z,k) \pm \overline{m^{12}(z,\bar{k})} \\ v_\pm(k) &= m^{21}(z,k) \pm \overline{m^{22}(z,\bar{k})}, \end{aligned}$$

then each of these functions satisfies an equation of the form $\bar{\partial} w = r\bar{w}$. Here, r lies in $L^2(\mathbf{R}^2)$ since S lies in L^2. This calculation relies on the symmetry $S(\bar{k}) = S(k)^*$ which is discussed in [3]. Using Theorem 2.3 we may conclude that $m(z,\cdot)$ lies in L^p for p close to infinity. Then Corollary 3.11 implies that each of the functions $u_\pm, v_\pm$ are zero for each z. Thus $m_1 = m_2$. ∎

Our next step is to observe that we may recover Q from the special solutions m.

Theorem 5.2 *Let Q be in L^p_c, $p > 2$, and let*

$$\psi = m \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}$$

be the special solutions constructed in Theorem A. Then for any $r > 0$, we have

$$Q(z) = \lim_{k_0 \to \infty} \mu(B_r(0))^{-1} \int_{\{k : |k - k_0| < r\}} D_k m(z,k)\, d\mu(k).$$

Proof. We recall that m satisfies the equation

$$D_k m - Qm = 0.$$

From Theorem 2.3, we observe that $m - 1 \in L^\infty_z(L^q_k)$ for some $q < \infty$. This implies that

$$\begin{aligned}\lim_{k_0\to\infty}\int_{\{k:|k-k_0|<r\}} D_k m(z,k)\,d\mu(k) &= Q(z)\lim_{k_0\to\infty}\int_{\{k:|k-k_0|<r\}} m(z,k)\,d\mu(k)\\ &= \mu(B_r(0))Q(z).\end{aligned}$$

∎

Finally, the solution of the inverse boundary value problem follows easily from the above uniqueness result for the scattering problem

Corollary 5.3 *Suppose γ_1 and γ_2 are two conductivities in $W^{1,p}(\Omega)$, $p > 2$, and Ω is a Lipschitz domain. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$.*

Proof. Since $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, we have $\gamma_1 = \gamma_2$ on $\partial\Omega$. Thus we may extend γ_1 and γ_2 to $\mathbf{R}^2\backslash\bar{\Omega}$ so that $\gamma_1 - \gamma_2 = 0$ in $\mathbf{R}^2\backslash\bar{\Omega}$, $\gamma_1 = \gamma_1 = 1$ near infinity and $\nabla\gamma_j$ are in $L^p_c(\mathbf{R}^2)$. As in Theorem 4.1 we have that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $S_1 = S_2$. From Theorem 5.2, we obtain that $\partial\log(\gamma_1/\gamma_2) = \bar{\partial}\log(\gamma_1/\gamma_2) = 0$. Since $\gamma_1 = \gamma_2 = 1$ near infinity, this implies $\gamma_1 = \gamma_2$. ∎

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References

[1] Giovanni Alessandrini. Singular solutions of elliptic equations and the determination of conductivity by boundary measurements. *J. Diff. Equations*, 84:252–272, 1990.

[2] R. Beals and R.R. Coifman. Multidimensional inverse scatterings and nonlinear partial differential equations. In F. Treves, editor, *Pseudodifferential operators and applications*, volume 43 of *Proceedings of symposia in pure mathematics*, pages 45–70. Amer. Math. Soc., 1985.

[3] R. Beals and R.R. Coifman. The spectral problem for the Davey-Stewartson and Ishimori hierarchies. In *Nonlinear evolution equations: Integrability and spectral methods*, pages 15–23. Manchester University Press, 1988.

[4] Russell M. Brown. Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result. Submitted to *SIAM J. Math. Anal.*

[5] Russell M. Brown. Global uniqueness in the impedance imaging problem for less regular conductivities. *SIAM J. Math. Anal.*, 27:1049–1056, 1996.

[6] S. Chanillo. A problem in electrical prospection and a n-dimensional Borg-Levinson theorem. *Proc. Amer. Math. Soc.*, pages 761–767, 1990.

[7] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order.* Springer-Verlag, Berlin, 1983.

[8] Victor Isakov. On uniqueness of recovery of a discontinuous conductivity coefficient. *Comm. Pure Appl. Math.*, 41:865–977, 1988.

[9] R.V. Kohn and Michael Vogelius. Determining conductivity by boundary measurements, II, interior results. *Comm. Pure Appl. Math.*, 38:643–667, 1985.

[10] Adrian I. Nachman. Reconstructions from boundary measurements. *Annals of Math.*, 128:531–587, 1988.

[11] Adrian I. Nachman. Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math. (2)*, 143:71–96, 1996.

[12] Adrian I. Nachman, J. Sylvester, and G. Uhlmann. An n-dimensional Borg-Levinson theorem. *Comm. Math. Physics*, 115:595–605, 1988.

[13] R.G. Novikov. Multidimensional inverse spectral problem for the equation $-\Delta\psi+(v(x)-Eu(x))\psi = 0$. *Functional anal. and appl.*, 22:263–272, 1988.

[14] E.M. Stein. *Harmonic Analysis: Real-variable methods, orthogonality and oscillatory integrals.* Princeton, 1993.

[15] Li-yeng Sung. An inverse scattering transform for the Davey-Stewartson II equations, I. *J. Math. Anal. Appl.*, 183:121–154, 1994.

[16] Li-yeng Sung. An inverse scattering transform for the Davey-Stewartson II equations, II. *J. Math. Anal. Appl.*, 183:289–325, 1994.

[17] Li-yeng Sung. An inverse scattering transform for the Davey-Stewartson II equations, III. *J. Math. Anal. Appl.*, 183:477–494, 1994.

[18] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Annals of Math.*, 125:153–169, 1987.

[19] C. F. Tolmasky. Exponentially growing solutions for non-smooth first order perturbations of the Laplacian. preprint, 1996.

[20] I.N. Vekua. *Generalized Analytic functions.* Addison Wesley Pub. Co., Reading, Mass., 1962.

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