On the Dimension of the Attractor for the Non-Homogeneous Navier-Stokes Equations in Non-Smooth Domains

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Abstract

This paper concerns the two-dimensional Navier-Stokes equations in a Lipschitz domain Ω with nonhomogeneous boundary condition $u = \varphi$ on $\partial\Omega$. Assuming $\varphi \in L^\infty(\partial\Omega)$, we establish the existence of the universal attractor, and show that its dimension is bounded by $c_1 G + c_2 \mathrm{Re}^{3/2}$, where G is the Grashof number and Re the Reynolds number.

1 Introduction

Consider the two-dimensional Navier-Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \\ \\ \operatorname{div} u = 0 \end{cases} \qquad \text{in } \ \Omega \tag{1.1}$$

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with the nonhomogeneous boundary condition

$$u = \varphi \quad \text{on } \partial\Omega \tag{1.2}$$

where $f \in L^2(\Omega)$ and $\varphi \in L^\infty(\partial\Omega)$ are time-independent functions. We consider this equation in an appropriate Hilbert space and show that there is an attractor $\mathcal{A}$ which all solutions approach as $t \to \infty$. Furthermore, we show that this attractor has finite Hausdorff and fractal dimensions and establish that

$$\dim\mathcal{A} \leq c_1\, G + c_2\, \mathrm{Re}^{3/2} \tag{1.3}$$

where G is the Grashof number, Re is the Reynolds number, and c_1, c_2 are nondimensional constants depending on Ω. The main interest of this work lies in our assumptions on the domain Ω occupied by the fluid as well as on the nonhomogeneous boundary data φ. Indeed, we will only assume that Ω is a (simply connected) Lipschitz domain in $\mathbf{R}^2$ and

$$\varphi \in L^\infty(\partial\Omega), \quad \varphi \cdot n = 0 \quad \text{a.e. on } \partial\Omega \tag{1.4}$$

where n is the outward unit normal to $\partial\Omega$. Such assumptions are much more physically realistic than the ones in the existing estimates. In particular, our study covers the classical driven cavity model where $\Omega = (0,1) \times (0,1)$ is a square and $\varphi = (1,0)$ on $(0,1) \times \{1\}$, $\varphi = (0,0)$ otherwise.

The study of attractors for the Navier-Stokes equations has received considerable attention in recent years in an attempt to understand turbulence and chaos mathematically. In the case that Ω is smooth and $\varphi = 0$, the dimension estimate (1.3) reduces to $\dim\mathcal{A} \leq c_1\, G$ and is well-known. We refer the reader to [3, 4, 7] and Temam's monograph [16] for further references.

Recently, it was shown by A. Ilyin [8] that, if $\varphi = 0$, the estimate $\dim\mathcal{A} \leq c\, G$ in fact is valid for arbitrary domains in $\mathbf{R}^2$ with finite measures. For flows driven by boundary conditions, (1.3) was established by A. Miranville and X. Wang [9] under the assumptions that $\partial\Omega$ is C^3 and $|\nabla\varphi| \in L^\infty(\partial\Omega)$. The present work extends the result of Miranville and Wang to the nonsmooth setting.

The paper is organized as follows. In section 2, we reduce the problem (1.1)-(1.2) to equations similar to the Navier-Stokes equations with homogeneous boundary condition. This will be done by constructing a function ψ (background flow) such that

$$\operatorname{div}\psi = 0 \;\text{ in } \Omega \text{ and } \psi = \varphi \text{ on } \partial\Omega. \tag{1.5}$$

The basic idea of our construction, which is motivated by the work of Miranville-Wang, is to localize the solution of the Stokes system with boundary data φ to a ε-neighborhood of $\partial\Omega$. Let $v = u - \psi$ where u is a solution of (1.1)-(1.2). Then v satisfies, at least formally,

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + (v\cdot\nabla)\psi + (\psi\cdot\nabla)v + \nabla p = f + \nu\Delta\psi - (\psi\cdot\nabla)\psi \\ \operatorname{div} v = 0 \end{cases} \tag{1.6}$$

in Ω and

$$v = 0 \;\text{ on } \partial\Omega. \tag{1.7}$$

In section 3, we establish the existence and uniqueness of weak solutions to (1.6)-(1.7). We also give a definition of weak solution to the boundary value problem (1.1)-(1.2) for the Navier-Stokes equations. This definition is motivated by our construction of solutions as a sum of a background flow and a solution to equation (1.6)-(1.7). We prove that with our definition we have existence and uniqueness. It is easy to check that a sufficiently smooth solution of the Navier Stokes equations (1.1)-(1.2) also satisfies our definition.

In section 4 we show that, if $v|_{t=0} = v_0 \in \mathcal{H}$, then the solution of (1.6)-(1.7) satisfies

$$v \in L^\infty_{\text{loc}}((0,\infty);\mathcal{D}(A^{1/4})) \tag{1.8}$$

where

$$\mathcal{H} = \{f \in L^2(\Omega);\; \operatorname{div} f = 0 \;\text{ in } \Omega,\;\; f\cdot n = 0 \text{ on } \partial\Omega\} \tag{1.9}$$

and $\mathcal{D}(A^{1/4})$ denotes the domain of $A^{1/4}$, with A being the Stokes operator. We remark that, since φ is merely a bounded function on $\partial\Omega$, $\psi \notin H^1(\Omega)$ in general. Thus one may not expect $v \in L^\infty_{\text{loc}}((0,\infty);\mathcal{D}(A^{1/2}))$ as in the standard theory, even if the initial data v_0 is smooth.

Let $w(t) = S(t)v_0$ denote the solution to (1.6)-(1.7) with initial data v_0. Using estimates obtained in section 4, we show in section 5 that the semigroup $S(t)$ is uniformly differentiable on bounded subsets of $\mathcal{H}$, and the ball in $\mathcal{D}(A^{1/4})$ centered at 0 with a suitable radius absorbs any bounded set of $\mathcal{H}$. This, together with the abstract machinery in [16], gives the existence of the universal attractor as well as the desired estimate of its dimension.

To state the main theorem, we introduce the Grashof number G and the Reynolds number Re:

$$G = \frac{\|f\|_{L^2(\Omega)}}{\nu^2\lambda_1}, \qquad \mathrm{Re} = \frac{\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu\lambda_1^{1/2}}. \tag{1.10}$$

In the above, λ_1 is the first eigenvalue of the Stokes operator A.

The following is the main result of the paper.

Theorem 1.11 *Let Ω be a simply connected Lipschitz domain in $\mathbf{R}^2$. Suppose $\varphi \in L^\infty(\partial\Omega)$, $\varphi \cdot n = 0$, and $f \in L^2(\Omega)$. Then*

(i) The dynamical system associated to (1.6)–(1.7), more precisely, to the abstract differential equation (3.1), possesses an universal attractor $\mathcal{A}$,

(ii) The Hausdorff and fractal dimensions of $\mathcal{A}$ are bounded by $c_1G + c_2Re^{3/2} + 1$, where c_1, c_2 are nondimensional constants depending on Ω.

Remark 1.12. The background flow ψ in (1.5) is C^∞ in Ω and belongs to $H^{1/2}(\Omega) \cap L^\infty(\Omega)$. Also, $\psi = \varphi$ on $\partial\Omega$ in the sense of nontangential convergence. See Theorem 2.3, Propositions 2.13 and 2.14.

Remark 1.13. Note that if $S(t)v_0$ denotes the solution to (1.6)-(1.7) with the initial data v_0, then $\psi + S(t)(u_0 - \psi)$ is the solution to (1.1)-(1.2) with the initial data u_0

and boundary data φ. Hence the universal attractor for (1.1)-(1.2) is given by the translation $\psi + \mathcal{A} = \{\psi + v : \ v \in \mathcal{A}\}$.

2 Construction of Background Flow

Let Ω be a bounded domain in $\mathbf{R}^d$. We say that Ω is a Lipschitz domain if its boundary $\partial\Omega$ can be covered by finite many balls $B_j = B(Q_j, r_0)$ centered at $Q_j \in \partial\Omega$ such that for each B_j, there exists a rectangular coordinate system and a Lipschitz function $\psi_j : \ \mathbf{R}^{d-1} \to \mathbf{R}$ with

$$B(Q_j, 3r_0) \cap \Omega = \{(x_1, \cdots, x_d); \quad x_d > \psi_j(x_1, \cdots, x_{d-1})\} \cap \Omega.$$

Throughout this paper we will assume that Ω is a simply connected Lipschitz domain in $\mathbf{R}^2$.

For a function u on Ω, we define its nontangential maximal function $(u)^*$ by

$$(u)^*(Q) = \sup\{|u(x)|; \ x \in \Omega, \quad |x - Q| \le 2\,\mathrm{dist}(x, \partial\Omega)\}, \quad Q \in \partial\Omega. \tag{2.1}$$

As we mentioned in the introduction, our background flow will be constructed using the solution to the Stokes system:

$$\begin{cases} -\Delta u + \nabla q = 0 & \text{in } \Omega \\ \mathrm{div}\, u = 0 & \text{in } \Omega \\ u = \varphi \ \text{ a.e.} & \text{on } \partial\Omega \text{ in the sense of nontangential convergence.} \end{cases} \tag{2.2}$$

Theorem 2.3 *Let Ω be a simply connected Lipschitz domain in $\mathbf{R}^2$. If $\varphi \in L^2(\partial\Omega)$ and $\int_{\partial\Omega} \varphi \cdot n \, d\sigma = 0$, there exists a unique u and a unique (up to a constant) q satisfying (2.2) and $(u)^* \in L^2(\partial\Omega)$. In fact, the solution (u, q) will satisfy*

$$\int_{\partial\Omega} |(u)^*|^2 \, d\sigma + \int_\Omega |\nabla u(x)|^2 \, \mathrm{dist}(x, \partial\Omega) \, dx + \int_\Omega |q(x)|^2 \ \mathrm{dist}(x, \partial\Omega) \, dx \le C \int_{\partial\Omega} |\varphi|^2 \, d\sigma. \tag{2.4}$$

If, in addition, $\varphi \in L^\infty(\partial\Omega)$, then

$$\sup_{x \in \Omega} |u(x)| + \sup_{x \in \Omega} |\nabla u(x)| \, \mathrm{dist}(x, \partial\Omega) \le C \|\varphi\|_{L^\infty(\partial\Omega)}. \tag{2.5}$$

Remark. If Ω is a Lipschitz domain in $\mathbf{R}^d$, $d \geq 3$ with connected boundary, the L^2-estimate (2.4) was established in [6]. Also see [2]. In the case $d = 3$, the L^∞-estimate (2.5) was obtained in [12]. The arguments in [6] and [12] can be extended to the case $d = 2$, with some modifications. The two-dimensional case is slightly different because of the logarithmic singularity of the fundamental solution. In the appendix, we will indicate the changes that are needed in two dimensions.

Let $u = (u_1, u_2)$ be the solution of (2.2) with $\varphi \in L^\infty(\partial\Omega)$ and $\varphi \cdot n = 0$. Fix $P \in \partial\Omega$. We define

$$g(x) = \int_P^x (-u_2, u_1) \cdot T \, ds \tag{2.6}$$

where T denotes the unit tangent vector to the path from P to $x = (x_1, x_2)$. Since Ω is simply connected and $\operatorname{div} u = 0$ in Ω, g is well-defined by Green's theorem, and

$$u = \left(\frac{\partial g}{\partial x_2}, -\frac{\partial g}{\partial x_1} \right). \tag{2.7}$$

Moreover, since $u = \varphi$ on $\partial\Omega$ and $\varphi \cdot n = 0$ a.e., we have

$$g = 0 \quad \text{on } \partial\Omega. \tag{2.8}$$

Next let $\varepsilon \in (0, c \operatorname{diam}(\Omega))$ be a constant to be determined later. Let $\eta_\varepsilon \in C_0^\infty(\mathbf{R}^2)$ such that, $0 \leq \eta \leq 1$,

$$\begin{cases} \eta_\varepsilon = 1 & \text{in } \{x \in \mathbf{R}^2; \operatorname{dist}(x, \partial\Omega) \leq c_1 \varepsilon\} \\ \eta_\varepsilon = 0 & \text{in } \{x \in \mathbf{R}^2; \operatorname{dist}(x, \partial\Omega) \geq c_2 \varepsilon\} \end{cases} \tag{2.9}$$

and

$$|\nabla^\alpha \eta_\varepsilon| \leq c_\alpha / \varepsilon^{|\alpha|}. \tag{2.10}$$

We remark that η_ε can be found in the form $f\left(\frac{\rho(x)}{\varepsilon}\right)$ where $\rho \in C^\infty$ is a regularized distance function to $\partial\Omega$ (see [13, p.170]) and f is a standard bump function.

Finally, we define the background flow

$$\psi = \psi_\varepsilon = \left(\frac{\partial}{\partial x_2}(g\eta_\varepsilon), -\frac{\partial}{\partial x_1}(g\eta_\varepsilon) \right). \tag{2.11}$$

Clearly, $\operatorname{div}\psi = 0$ in Ω, $\psi = u$ in $\{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) < c_1\varepsilon\}$. Hence, $\psi = \varphi$ on $\partial\Omega$ in the sense of nontangential convergence. Also note that

$$\operatorname{supp}\psi \subset \{x \in \overline{\Omega};\ \operatorname{dist}(x, \partial\Omega) \le c_2\varepsilon\}. \tag{2.12}$$

Proposition 2.13 *With φ and ψ as above, we have*

$$\|\psi\|_{L^\infty(\Omega)} \le C\|\varphi\|_{L^\infty(\partial\Omega)}.$$

Proof. Note that, by (2.11), (2.7) and (2.5),

$$|\psi(x)| \le |\nabla g(x)| + |g(x)|\ |\nabla\eta_\varepsilon(x)| \le C\|\varphi\|_{L^\infty(\partial\Omega)} + |g(x)|\ |\nabla\eta_\varepsilon(x)|.$$

To estimate the second term, by (2.12), we may assume $\operatorname{dist}(x, \partial\Omega) \le c_2\,\varepsilon$. Since $g = 0$ on $\partial\Omega$, $|g(x)| \le C\varepsilon\|\nabla g\|_{L^\infty(\Omega)} = C\varepsilon\|u\|_{L^\infty(\Omega)}$. Thus, by (2.10) and (2.5),

$$|g(x)|\ |\nabla\eta_\varepsilon(x)| \le \frac{C}{\varepsilon}|g(x)| \le C\|\varphi\|_{L^\infty(\partial\Omega)}.$$

∎

Proposition 2.14 *Let $2 \le p \le \infty$. Then*

$$\|\ |\nabla\psi| \operatorname{dist}(\cdot, \Omega)^{1-1/p}\|_{L^p(\Omega)} \le C\|\varphi\|_{L^p(\partial\Omega)}.$$

Proof. It follows from (2.4), (2.5) and complex interpolation that

$$\|\ |\nabla u| \operatorname{dist}(\cdot, \partial\Omega)^{1-1/p}\|_{L^p(\Omega)} \le C\|\varphi\|_{L^p(\partial\Omega)}, \quad 2 \le p \le \infty. \tag{2.15}$$

Note that, by the definition (2.11) of ψ and (2.10),

$$|\nabla\psi| \le C\{|\nabla u| + \frac{1}{\varepsilon}|u| + \frac{1}{\varepsilon^2}|g|\}.$$

With (2.15), we only need to estimate $\frac{1}{\varepsilon}|u|$ and $\frac{1}{\varepsilon^2}|g|$. We may assume $\operatorname{dist}(x, \partial\Omega) \le c_2\,\varepsilon$ in view of (2.12).

For $\frac{1}{\varepsilon}|u|$, we note that

$$\begin{aligned}
&\int_{\operatorname{dist}(x,\partial\Omega)\le c_2\varepsilon} \left|\frac{u}{\varepsilon}\right|^p \operatorname{dist}(x,\partial\Omega)^{p-1}\, dx \\
&\le \frac{C}{\varepsilon} \int_{\operatorname{dist}(x,\partial\Omega)\le c_2\varepsilon} |u|^p\, dx \le C\int_{\partial\Omega} |(u)^*|^p\, d\sigma \le C\int_{\partial\Omega} |\varphi|^p\, d\sigma
\end{aligned}$$

where the last inequality is a consequence of (2.4)-(2.5) and real interpolation.

Similarly,

$$\begin{aligned}
\int_{\operatorname{dist}(x,\partial\Omega)\le c_2\varepsilon} \left|\frac{g}{\varepsilon^2}\right|^p \operatorname{dist}(x,\partial\Omega)^{p-1}\, dx &\le \frac{C}{\varepsilon^{p+1}} \int_{\operatorname{dist}(x,\partial\Omega)\le c_2\varepsilon} |g|^p\, dx \\
&\le \frac{C}{\varepsilon^p} \int_{\partial\Omega} |(g)^*_\varepsilon|^p\, d\sigma
\end{aligned}$$

where

$$(g)^*_\varepsilon(Q) = \sup \{|g(x)|;\ x\in\Omega,\ \operatorname{dist}(x,\partial\Omega)\le c_2\,\varepsilon,\ \ |x-Q|<2\operatorname{dist}(x,\partial\Omega)\}.$$

Since for any $x\in\Omega$ with $\operatorname{dist}(x,\partial\Omega)\le c_2\,\varepsilon$ and $|x-Q|<2\operatorname{dist}(x,\partial\Omega)$,

$$|g(x)| = |g(x)-g(Q)| \le C\varepsilon\,(\nabla g)^*(Q) = C\varepsilon\,(u)^*(Q),$$

we have $(g)^*_\varepsilon(Q)\le C\varepsilon\,(u)^*(Q)$. It follows that

$$\int_{\operatorname{dist}(x,\partial\Omega)\le c_2\varepsilon} \left|\frac{g}{\varepsilon^2}\right|^p \operatorname{dist}(x,\partial\Omega)^{p-1}\, dx \le C\int_{\partial\Omega} |(u)^*|^p\, d\sigma \le C\int_{\partial\Omega} |\varphi|^p\, d\sigma.$$

The proof is complete. ∎

Proposition 2.16 *Let ψ be defined by (2.11). Then*

$$\Delta\psi = \nabla(q\eta_\varepsilon) + F$$

where $\operatorname{supp} F \subset \{x\in\Omega;\ c_1\,\varepsilon \le \operatorname{dist}(x,\partial\Omega)\le c_2\,\varepsilon\}$ *and*

$$\|F\|_{L^2(\Omega)} \le \frac{C}{\varepsilon^{3/2}} \|\varphi\|_{L^2(\partial\Omega)}.$$

Proof. A simple computation shows

$$\Delta\psi = \nabla(q\eta_\varepsilon) + F \tag{2.17}$$

where

$$\begin{aligned} F = q\nabla\eta_\varepsilon + 2\nabla\eta_\varepsilon\cdot\nabla u + u\Delta\eta_\varepsilon + \Delta g\left(\frac{\partial\eta_\varepsilon}{\partial x_2}, -\frac{\partial\eta_\varepsilon}{\partial x_1}\right) \\ +2\nabla g\cdot\nabla\left(\frac{\partial\eta_\varepsilon}{\partial x_2}, -\frac{\partial\eta_\varepsilon}{\partial x_1}\right) + g\Delta\left(\frac{\partial\eta_\epsilon}{\partial x_2}, -\frac{\partial\eta_\epsilon}{\partial x_1}\right). \end{aligned}$$

Clearly, $\operatorname{supp} F \subset \{x\in\Omega;\ c_1\,\varepsilon \le \operatorname{dist}(x,\partial\Omega) \le c_2\,\varepsilon\} = (\partial\Omega)_\varepsilon$ and

$$|F| \le C\left\{\left|\frac{q}{\varepsilon}\right| + \left|\frac{\nabla u}{\varepsilon}\right| + \left|\frac{u}{\varepsilon^2}\right| + \left|\frac{g}{\varepsilon^3}\right|\right\}.$$

It follows that

$$\int_\Omega |F|^2\,dx \le C\left\{\int_{(\partial\Omega)_\varepsilon}\left|\frac{q}{\varepsilon}\right|^2\,dx + \int_{(\partial\Omega)_\varepsilon}\left|\frac{\nabla u}{\varepsilon}\right|^2\,dx + \int_{(\partial\Omega)_\varepsilon}\left|\frac{u}{\varepsilon^2}\right|^2 dx + \int_{(\partial\Omega)_\varepsilon}\left|\frac{g}{\varepsilon^3}\right|^2\,dx\right\}.$$

Using (2.4), we have

$$\int_{(\partial\Omega)_\varepsilon}\left|\frac{q}{\varepsilon}\right|^2\,dx \le \frac{C}{\varepsilon^3}\int_\Omega |q|^2\operatorname{dist}(x,\partial\Omega)\,dx \le \frac{C}{\varepsilon^3}\int_{\partial\Omega}|\varphi|^2\,d\sigma,$$

$$\int_{(\partial\Omega)_\varepsilon}\left|\frac{\nabla u}{\varepsilon}\right|^2\,dx \le \frac{C}{\varepsilon^3}\int_\Omega |\nabla u|^2\operatorname{dist}(x,\partial\Omega)\,dx \le \frac{C}{\varepsilon^3}\int_{\partial\Omega}|\varphi|^2\,d\sigma,$$

$$\int_{(\partial\Omega)_\varepsilon}\left|\frac{u}{\varepsilon^2}\right|^2\,dx \le \frac{C}{\varepsilon^4}\int_{(\partial\Omega)_\varepsilon}|u|^2\,dx \le \frac{C}{\varepsilon^3}\int_{\partial\Omega}|(u)^*|^2\,d\sigma \le \frac{C}{\varepsilon^3}\int_{\partial\Omega}|\varphi|^2\,d\sigma.$$

Finally,

$$\begin{aligned} \int_{(\partial\Omega)_\varepsilon}\left|\frac{g}{\varepsilon^3}\right|^2\,dx &= \frac{1}{\varepsilon^6}\int_{(\partial\Omega)_\varepsilon}|g|^2\,dx \\ &\le \frac{C}{\varepsilon^5}\int_{\partial\Omega}|(g)^*_\varepsilon|^2\,d\sigma \le \frac{C}{\varepsilon^3}\int_{\partial\Omega}|(\nabla g)^*_\varepsilon|^2\,d\sigma \\ &\le \frac{C}{\varepsilon^3}\int_{\partial\Omega}|(u)^*|^2\,d\sigma \le \frac{C}{\varepsilon^3}\int_{\partial\Omega}|\varphi|^2\,d\sigma, \end{aligned}$$

where we have used $(g)^*_\varepsilon \le C\varepsilon(\nabla g)^*_\varepsilon$. The estimate of $\|F\|_{L^2(\Omega)}$ now follows. ∎

We now set $v = u - \psi$ where u is a solution of (1.1). Using (2.17), formally we have

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla) v + (v \cdot \nabla)\psi + (\psi \cdot \nabla) v \\ \qquad\qquad + \nabla(p + \nu q \eta_\varepsilon) = f + \nu F - (\psi \cdot \nabla)\psi \\ \operatorname{div} v = 0 \\ v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.18}$$

3 Existence of Weak Solutions to (2.18)

We begin with a list of notation.

$\mathcal{H} = \{u \in L^2(\Omega);\ \operatorname{div} u = 0 \ \text{ in } \Omega,\ u \cdot n = 0 \quad \text{on } \partial\Omega\}$,

$V = \{u \in H_0^1(\Omega);\ \operatorname{div} u = 0 \ \text{ in } \Omega\}$,

$|\cdot|_p$, the $L^p(\Omega)$ norm,

$\|\cdot\|$, the norm in V,

$\langle\ ,\ \rangle$, the inner product in $\mathcal{H}$ or the dual product between V and V',

$(\ ,\ )$ the inner product in V.

We let A denote the Stokes operator, which may be defined as the unique positive self-adjoint operator associated with the quadratic form $(\cdot,\cdot)$ on V (see [10, Theorem VIII.15]). We let $B(u,v) = (u \cdot \nabla)v$, and we will see below that this defines an element of $H^{-1}(\Omega) \subset V'$. We let P be the orthogonal projector in $L^2(\Omega)$ on the space $\mathcal{H}$. In view of (2.18), we consider the differential equation

$$\begin{cases} \dfrac{dv}{dt} + \nu A v + B(v,v) + B(v,\psi) + B(\psi,v) = P(f + \nu F) - B(\psi,\psi) \\ v(0) = v_0 \in \mathcal{H}. \end{cases} \tag{3.1}$$

We point out that $\langle B(v,\psi),w\rangle$, $\langle B(\psi,v),w\rangle$ and $\langle B(\psi,\psi),w\rangle$ are well-defined if $v,w \in V$. This follows easily from the estimate

$$|\psi(x)| + |\nabla\psi(x)| \ \mathrm{dist}(x,\partial\Omega) \le C\,\|\varphi\|_{L^\infty(\partial\Omega)} \tag{3.2}$$

(see Propositions 2.13 and 2.14) and Hardy's inequality

$$\int_\Omega \frac{|v(x)|^2}{[\mathrm{dist}(x,\partial\Omega)]^2}\,dx \le C\int_\Omega |\nabla v(x)|^2\,dx, \quad \text{for } v \in H_0^1(\Omega). \tag{3.3}$$

Thus (3.1) is an abstract differential equation in V'.

We first establish the existence of solutions of (3.1) by the standard Faedo-Galerkin method.

Let $\{w_j\}_{j=1}^\infty$ be an orthonormal basis of $\mathcal{H}$ such that $Aw_j = \lambda_j w_j, \quad \lambda_1 \le \lambda_2 \le \cdots$. Fix $m \ge 1$, let

$$v_m(t) = \sum_{j=1}^m g_{jm}(t) w_j.$$

We solve the system of ODE's

$$\begin{cases} \left\langle \dfrac{dv_m}{dt},\ w_j \right\rangle + \nu(v_m,w_j) + b(v_m,v_m,w_j) + b(\psi,v_m,w_j) + b(v_m,\psi,w_j) \\ \qquad = \langle \bar f, w_j\rangle - b(\psi,\psi,w_j), \quad j=1,2,\cdots,m, \\ v_m(0) = P_m v_0 \end{cases} \tag{3.4}$$

where $b(u,v,w) = \langle B(u,v),w\rangle$, $\quad \bar f = P(f+\nu F)$, and $P_m:\ \mathcal{H} \to \mathrm{span}\{w_1,\cdots,w_m\}$ is the projector.

We now show that $\{v_m(t)\}$ is a bounded set in $L^\infty((0,T);\mathcal{H}) \cap L^2((0,T);V)$ and $\left\{\dfrac{dv_m}{dt}\right\}$ is a bounded set in $L^2((0,T);V')$. By (3.4), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\,|v_m|_2^2 + \nu\|v_m\|^2 + b(v_m,v_m,v_m) + b(v_m,\psi,v_m) + b(\psi,v_m,v_m) \\ = \ \langle \bar f, v_m\rangle - b(\psi,\psi,v_m). \end{aligned}$$

Since $b(v_m,v_m,v_m)=0,\ b(\psi,v_m,v_m)=0$, we get

$$\frac{1}{2}\frac{d}{dt}|v_m|_2^2 + \nu\|v_m\|^2 \le |b(v_m,\psi,v_m)| + |\langle \bar f, v_m\rangle| + |b(\psi,\psi,v_m)|. \tag{3.5}$$

We estimate each term on the right-hand side of (3.5) separately. First we use (3.2), (2.12) and (3.3) to obtain

$$\begin{aligned}
|b(v_m,\psi,v_m)| &\le \int_\Omega |v_m|\;|\nabla\psi|\;|v_m|\,dx \\
&\le C|\varphi|_{L^\infty(\partial\Omega)}\int_{\mathrm{dist}(x,\partial\Omega)\le c_2\varepsilon}|v_m|^2\,\frac{dx}{\mathrm{dist}(x,\partial\Omega)} \\
&\le C\varepsilon|\varphi|_{L^\infty(\partial\Omega)}\int_\Omega\frac{|v_m|^2\,dx}{[\mathrm{dist}(x,\partial\Omega)]^2} \\
&\le C\varepsilon|\varphi|_{L^\infty(\partial\Omega)}\|v_m\|^2.
\end{aligned} \tag{3.6}$$

Choose

$$\varepsilon = c\cdot\min\left(\frac{\nu}{|\varphi|_{L^\infty(\partial\Omega)}},\quad \mathrm{diam}\,\Omega\right) \tag{3.7}$$

and c is so small that

$$|b(v_m,\psi,v_m)| \le \frac{\nu}{4}\|v_m\|^2. \tag{3.8}$$

Next, note that

$$|\langle\bar f,v_m\rangle| \le |\langle f,v_m\rangle| + \nu|\langle F,v_m\rangle| \le |f|_2\,|v_m|_2 + \nu\int_{c_1\varepsilon\le\mathrm{dist}(x,\partial\Omega)\le c_2\varepsilon}|F|\;|v_m|\,dx$$

since $\mathrm{supp}\,F\subset\{x\in\Omega;\ c_1\,\varepsilon\le\mathrm{dist}(x,\partial\Omega)\le c_2\,\varepsilon\}$. It then follows from Proposition 2.16 and (3.3) that

$$\begin{aligned}
|\langle\bar f,v_m\rangle| &\le |f|_2\cdot\frac{\|v_m\|}{\sqrt{\lambda_1}} + \nu|F|_2\cdot\left\{\int_\Omega\frac{|v_m|^2}{[\mathrm{dist}(x,\partial\Omega)]^2}\,dx\right\}^{1/2}\cdot c\,\varepsilon \\
&\le |f|_2\cdot\frac{\|v_m\|}{\sqrt{\lambda_1}} + \nu\cdot\frac{\|\varphi\|_{L^2(\partial\Omega)}}{\varepsilon^{3/2}}\cdot\|v_m\|\cdot c\,\varepsilon \\
&= \|v_m\|\left\{\frac{|f|_2}{\sqrt{\lambda_1}} + \frac{c\,\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)}\right\}.
\end{aligned} \tag{3.9}$$

Finally, by (3.2)-(3.3) and (2.12),

$$\begin{aligned}
&|b(\psi,\psi,v_m)| \le \int_\Omega|\psi|\;|\nabla\psi|\;|v_m|\,dx \le C\,\|\varphi\|_{L^\infty(\partial\Omega)}\int_\Omega\frac{|v_m|}{\mathrm{dist}(x,\partial\Omega)}|\psi|\,dx \\
&\le C\|\varphi\|_{L^\infty(\partial\Omega)}\cdot\left\{\int_\Omega\frac{|v_m|^2}{[\mathrm{dist}(x,\partial\Omega)]^2}\,dx\right\}^{1/2}\left\{\int_{\mathrm{dist}(x,\partial\Omega)\le c_2\varepsilon}|\psi|^2\,dx\right\}^{1/2} \\
&\le C\|\varphi\|^2_{L^\infty(\partial\Omega)}\,|\partial\Omega|^{1/2}\|v_m\|\cdot\sqrt{\varepsilon}.
\end{aligned}$$

This, together with (3.5), (3.8) and (3.9), gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v_m|_2^2 + \nu\|v_m\|^2 \\
&\quad \le \frac{\nu}{4}\|v_m\|^2 + \|v_m\| \left\{ \frac{|f|_2}{\sqrt{\lambda_1}} + \frac{c\,\nu}{\sqrt{\varepsilon}}\,\|\varphi\|_{L^2(\partial\Omega)} + C\sqrt{\varepsilon}|\partial\Omega|^{1/2}\,\|\varphi\|^2_{L^\infty(\partial\Omega)} \right\} \\
&\quad \le \frac{\nu}{2}\,\|v_m\|^2 + \frac{C}{\nu}\left\{ \frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}\,\|\varphi\|^2_{L^\infty(\partial\Omega)} \right\}^2
\end{aligned}$$

where we also used the Cauchy inequality in the second inequality.

It follows that

$$\frac{d}{dt}|v_m|_2^2 + \nu\|v_m\|^2 \le \frac{C}{\nu}\left\{ \frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}\|\varphi\|^2_{L^\infty(\partial\Omega)} \right\}^2. \tag{3.10}$$

Using (3.10), $\|v_m\|^2 \ge \lambda_1|v_m|_2^2$ and Gronwall's inequality, we then obtain

$$\begin{aligned}
&|v_m(t)|_2^2 \\
&\le e^{-\nu\lambda_1 t}|v_0|_2^2 + \frac{C}{\nu^2\lambda_1}\left\{ \frac{|f|_2}{\lambda_1} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\;|\partial\Omega|^{1/2}\;\|\varphi\|^2_{L^\infty(\partial\Omega)} \right\}^2.
\end{aligned} \tag{3.11}$$

By integration, (3.10) also gives

$$\begin{aligned}
&\nu\int_s^t \|v_m(\tau)\|^2\,d\tau \\
&\le\quad |v_m(s)|^2 + \frac{C}{\nu}\left\{ \frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}\;\|\varphi\|^2_{L^\infty(\partial\Omega)} \right\}^2 \cdot (t-s).
\end{aligned} \tag{3.12}$$

The estimates (3.11) and (3.12) show that $\{v_m(t)\}$ is a bounded set in $L^\infty((0,T);\mathcal{H})\cap L^2((0,T);V)$. By considering $\left\langle \frac{dv_m}{dt}, w \right\rangle$ for $w \in V$, we may prove that $\left\{\frac{dv_m}{dt}\right\}$ is a bounded set in $L^2((0,T);V')$ in a similar manner. This and the standard techniques found in [16] now give the following result.

Theorem 3.13 *Let $f \in \mathcal{H}$, $v_0 \in \mathcal{H}$. Then there exists a unique $v(t)$ such that $v(0) = v_0$,*

$$v \in C([0,T];\mathcal{H}) \cap L^2((0,T);V), \quad \frac{dv}{dt} \in L^2((0,T);V'), \quad \forall\; T > 0$$

and for any $w \in V$,

$$\begin{aligned}
\left\langle \frac{dv}{dt}, w \right\rangle + \nu(v(t),w) + b(v(t),v(t),w) + b(v(t),\psi,w) + b(\psi,v(t),w) \\
= \quad \langle \bar{f}, w\rangle - b(\psi,\psi,w) \quad a.e.\ t.
\end{aligned}$$

Now we give our definition of weak solution to the boundary value problem (1.1)-(1.2) for the Navier-Stokes equations. We note that with this definition of solution, we have existence and uniqueness. It is easy to see that a smooth solution of Navier-Stokes equations satisfies our definition.

Definition of Weak Solution. Let u_0 and f lie in the space $\mathcal{H}$. Let $\varphi \in L^\infty(\partial\Omega)$ and $\varphi \cdot n = 0$ on $\partial\Omega$. We say that u is a weak solution of the equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla u) = f - \nabla p, & \text{in } \Omega \times (0,T) \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0,T) \\ u = \varphi & \text{on } \partial\Omega \\ u(\cdot, 0) = u_0 \end{cases} \tag{3.14}$$

if the following three conditions hold:

(1) $u \in C([0,T]; \mathcal{H})$, $u(\cdot,0) = u_0$, and $du/dt \in L^2_{loc}((0,T); V')$.

(2) For every $v \in C_0^\infty(\Omega)$ with $\operatorname{div} v = 0$, we have

$$\frac{d}{dt}\langle u, v\rangle - \nu \langle u, \Delta v\rangle - \int_\Omega u^i u^j \frac{\partial v^i}{\partial x_j}\, dx = \langle f, v\rangle$$

as distributions on $(0,T)$. Here, we are using the summation convention.

(3) There exist functions $\psi \in C^2(\Omega) \cap L^\infty(\Omega)$, $q \in C^1(\Omega)$ and $g \in L^2(\Omega)$ so that

$$\begin{cases} \Delta \psi = \nabla q + g & \text{in } \Omega \\ \operatorname{div} \psi = 0 & \text{in } \Omega \\ \psi = \varphi & \text{on } \partial\Omega. \end{cases}$$

We assume that ψ obtains its boundary values in the sense of nontangential convergence as in [6]. Finally, we require that the function $u - \psi$ lie in $L^2((0,T); V)$.

Remark 1. We first observe that if we have two background flows, ψ_1 and ψ_2 as in (3), then $\psi_1 - \psi_2 \in V$. To see this, observe that we can use the Lax-Milgram Theorem to construct a solution of

$$\begin{cases} \Delta w = g_1 - g_2 + \nabla(q_1 - q_2) & \text{in } \Omega \\ \operatorname{div} w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

which lies in V. By the estimate (A.12) for the solutions of the Stokes system in the appendix, one must have $w = \psi_1 - \psi_2$. Additional arguments also give that $\psi_1 - \psi_2$ lies in $H^{3/2}(\Omega)$ (see [2]), but we do not need this.

Remark 2. If u is a solution as defined above, then we also have that $u \in L^4(\Omega \times (0,T))$. In fact, we have

$$\begin{aligned}\left(\int_\Omega |u(x,t)|^4\,dx\right)^{1/4} &\leq \left(\int_\Omega |u-\psi|^4\,dx\right)^{1/4} + \left(\int_\Omega |\psi|^4\,dx\right)^{1/4} \\ &\leq C\left(\int_\Omega |\nabla(u-\psi)|^2\,dx\right)^{1/4}\left(\int_\Omega |u-\psi|^2\,dx\right)^{1/4} + \left(\int_\Omega |\psi|^4\,dx\right)^{1/4}.\end{aligned}$$

Now we use that $\psi \in L^\infty(\Omega)$, $u - \psi \in L^2((0,T);V)$ and $u \in C([0,T];\mathcal{H})$.

Theorem 3.15 *Let $u_0 \in \mathcal{H}$, $f \in \mathcal{H}$. Suppose that $\varphi \in L^\infty(\partial\Omega)$ and $\varphi \cdot n = 0$ on $\partial\Omega$. Then (3.14) has a unique weak solution.*

Proof. We begin with the uniqueness. Suppose that u_1 and u_2 are two solutions with associated flows ψ_1 and ψ_2. Let $v \in C_0^\infty(\Omega)$ and $\operatorname{div} v = 0$. Then by (2) we have

$$\frac{d}{dt}\langle u_1 - u_2, v\rangle - \nu\langle u_1 - u_2, \Delta v\rangle + \int_\Omega (u_2^i u_2^j - u_1^i u_1^j)\frac{\partial v^i}{\partial x_j}\,dx = 0. \qquad (3.16)$$

We claim that we also have (3.16) for any $v \in V$. In fact, by Remark 1 and (3) of our definition of weak solution, we have $u_1 - u_2 = (u_1 - \psi_1) - (u_2 - \psi_2) + (\psi_1 - \psi_2) \in L^2((0,T);V)$. Thus we can write

$$\langle u_1 - u_2, \Delta v\rangle = -(u_1 - u_2, v).$$

We also have that for $\ell = 1,2$,

$$\left|\int_\Omega u_\ell^i u_\ell^j \frac{\partial v^i}{\partial x_j}\,dx\right| \leq \left(\int_\Omega |u_\ell|^4\,dx\right)^{1/2}\left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2}.$$

Hence, by Remark 2, we may take $v \in V$ in the second and third terms in (3.16). We conclude that

$$\frac{d}{dt}(u_1 - u_2) \in L^2((0,T);V')$$

and (3.16) holds for any $v \in V$. Let $v = u_1 - u_2$, then we obtain that

$$\frac{1}{2}\frac{d}{dt}|u_1 - u_2|^2 + \nu\|u_1 - u_2\|^2 \leq \left|\int_\Omega (u_1^i u_1^j - u_2^i u_2^j)\frac{\partial v^i}{\partial x_j}\, dx\right|.$$

Note that

$$\begin{aligned}\int_\Omega (u_1^i u_1^j - u_2^i u_2^j)\frac{\partial v^i}{\partial x_j}\, dx &= \int_\Omega u_1^i v^j \frac{\partial v^i}{\partial x^j} + u_2^j \frac{1}{2}\frac{\partial}{\partial x_j}|v|^2\, dx \\ &= \int_\Omega u_1^i v^j \frac{\partial v^i}{\partial x^j}\, dx,\end{aligned}$$

where the second equality may be justified by using $u_2 = (u_2 - \psi_2) + \psi_2$ and $\psi_2 \in L^\infty(\Omega)$, $u_2 - \psi_2 \in L^2((0,T), V)$. It follows that

$$\begin{aligned}\left|\int_\Omega (u_1^i u_1^j - u_2^i u_2^j)\frac{\partial v^i}{\partial x_j}\, dx\right| &\leq \left(\int_\Omega |u_1|^4\right)^{1/4}\left(\int_\Omega |v|^2\, dx\right)^{1/4}\left(\int_\Omega |\nabla v|^2\, dx\right)^{3/4} \\ &\leq \nu\|v\|^2 + C_\nu \int_\Omega |u_1|^4\, dx \int_\Omega |v|^2\, dx.\end{aligned}$$

Thus, we obtain the differential inequality

$$\frac{1}{2}\frac{d}{dt}|v|^2 \leq C_\nu |v|^2 \int_\Omega |u_1|^4\, dx.$$

Since we have that $u_1 \in L^4(\Omega \times (0,T))$ and $v(\cdot, 0) = 0$, this implies that $v = 0$ and we have established the uniqueness of our solutions.

Next, we establish existence of solutions. The main work has already been done in Proposition 2.16 and Theorem 3.13 where we constructed the family of background flows ψ_ϵ and the function v which satisfies (3.1). Now let $u = v + \psi_\epsilon$ where v is the solution of (3.1) with initial data $v_0 = u_0 - \psi_\varepsilon$ given in Theorem 3.13. It is easy to check that u satisfies the conditions (1) and (3) in the definition of the weak solution. To see (2), we observe that ψ_ϵ is in $C^\infty(\Omega)$ while the test function v is compactly supported. Thus the formal manipulations used to arrive at the equation (2.18) for v are easily justified to show that u satisfies (2). ∎

4 Regularity of weak solutions

We devote this section to the proof of $v \in L^\infty_{\mathrm{loc}}((0,T);\mathcal{D}(A^{1/4}))$ where v is the solution of (3.1) given in Theorem 3.13. Since $\varphi \in L^\infty(\partial\Omega)$, we cannot expect the solution $v \in L^\infty_{\mathrm{loc}}((0,T);\mathcal{D}(A^{1/2}))$.

Recall that the powers of the Stokes operator A are defined for $z \in \mathbf{C}$ by

$$A^z f = \sum_j \lambda_j^z a_j w_j \quad \text{for} \quad f = \sum_j a_j w_j$$

and

$$\mathcal{D}(A^z) = \{f;\ A^z f \in \mathcal{H}\} = \{f = \sum a_j w_j;\quad \sum_j \lambda_j^{2\mathrm{Re}z} |a_j|^2 < \infty\}.$$

Lemma 4.1 *There exists a constant C such that*

$$\int_\Omega \frac{|u(x)|^2\,dx}{\mathrm{dist}(x,\partial\Omega)} \le C \int_\Omega |A^{1/4}u|^2\,dx \qquad \textit{for } u \in \mathcal{D}(A^{1/4}).$$

Proof. Consider the operator

$$T_z = (\mathrm{dist}(x,\partial\Omega))^{-z} A^{-z/2}.$$

If $\mathrm{Re}z = 1$,

$$\int_\Omega |T_z u|^2\,dx = \int_\Omega \frac{|A^{-z/2}u|^2\,dx}{[\mathrm{dist}(x,\partial\Omega)]^2} \le C \int_\Omega |u|^2\,dx, \quad \text{for } u \in \mathcal{H}.$$

This follows from the fact $A^{-z/2}u \in \mathcal{D}(A^{1/2}) = V \subset H^1_0(\Omega)$ and (3.3). Clearly, if $\mathrm{Re}z = 0$,

$$\int_\Omega |T_z u|^2\,dx = \int_\Omega |u|^2\,dx \quad \text{for } u \in \mathcal{H}.$$

Thus, by Stein's interpolation theorem [14, p. 205], we obtain

$$\int_\Omega |T_{1/2}u|^2\,dx \le C \int_\Omega |u|^2\,dx,$$

i.e.

$$\int_\Omega \frac{|A^{-1/4}u|^2}{\mathrm{dist}(x,\partial\Omega)}\,dx \le C \int_\Omega |u|^2\,dx \quad \text{for } u \in \mathcal{H}.$$

Hence, if $u \in \mathcal{D}(A^{1/4})$,

$$\int_\Omega \frac{|u(x)|^2\,dx}{\mathrm{dist}(x,\partial\Omega)} \le C \int_\Omega |A^{1/4}u|^2\,dx.$$

∎

Remark. It follows from Lemma 4.1 that

$$\int_\Omega \frac{|A^\alpha u|^2}{\operatorname{dist}(x,\partial\Omega)}\,dx \le C\int_\Omega |A^{\alpha+1/4}u|^2\,dx \quad \text{if } u\in\mathcal{D}(A^{\alpha+1/4}). \tag{4.2}$$

Lemma 4.3 *There exists a constant C such that*

$$|u|_4 \le C|A^{1/4}u|_2 \quad \textit{for } u\in\mathcal{D}(A^{1/4}).$$

The constant C does not depend on Ω.

Proof. By Sobolev imbedding and a rescaling argument

$$\|u\|_{L^4(\mathbf{R}^2)} \le C\|(-\Delta)^{1/4}u\|_{L^2(\mathbf{R}^2)} \quad \text{if } (1+|\xi|)^{1/4}\hat{u}(\xi)\in L^2(\mathbf{R}^2).$$

Let E be the extension operator defined by

$$Eu = \begin{cases} u & x\in\Omega \\ 0 & x\notin\Omega. \end{cases}$$

Then

$$(-\Delta)^{z/2}E: \quad H_0^1(\Omega)\to L^2(\mathbf{R}^2) \qquad \text{if } \mathrm{Re}z=1,$$

$$(-\Delta)^{z/2}E: \quad L^2(\Omega)\to L^2(\mathbf{R}^2) \qquad \text{if } \mathrm{Re}z=0.$$

By interpolation,

$$(-\Delta)^{1/4}E: \ [H_0^1(\Omega),L^2(\Omega)]_{1/2}\to L^2(\mathbf{R}^2)$$

where $[\cdot,\cdot]_\theta$ denotes the complex interpolation space (see [1]). Thus, if $u\in\mathcal{D}(A^{1/4})$,

$$\|u\|_{L^4(\Omega)} \le C\|(-\Delta)^{1/4}Eu\|_{L^2(\mathbf{R}^2)} \le C\|u\|_{[H_0^1(\Omega),L^2(\Omega)]_{1/2}} \le C|A^{1/4}u|_{L^2(\Omega)}$$

where we have used $\mathcal{D}(A^{1/4}) = [\mathcal{D}(A^{1/2}),\mathcal{H}]_{1/2} \subset [H_0^1(\Omega),L^2(\Omega)]_{1/2}$ in the last inequality. ∎

We are now ready to estimate $\sup_{t \geq t_0 > 0} |A^{1/4} v(t)|_2$ where $v(t)$ is a solution given in Theorem 3.13. By (3.4),

$$\begin{aligned}
\left\langle \frac{dv_m}{dt}, A^{1/2} v_m \right\rangle + \nu \langle A v_m, A^{1/2} v_m \rangle + b(v_m, v_m, A^{1/2} v_m) & \\
+ b(\psi, v_m, A^{1/2} v_m) + b(v_m, \psi, A^{1/2} v_m) & \\
= \quad & \langle \bar{f}, A^{1/2} v_m \rangle + b(\psi, \psi, A^{1/2} v_m).
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} |A^{1/4} v_m|_2^2 + \nu |A^{3/4} v_m|_2^2 \leq |b(v_m, v_m, A^{1/2} v_m)| & \\
+ |b(\psi, v_m, A^{1/2} v_m)| + |b(v_m, \psi, A^{1/2} v_m)| & \qquad (4.4) \\
+ |\langle \bar{f}, A^{1/2} v_m \rangle| + |b(\psi, \psi, A^{1/2} v_m)|. &
\end{aligned}$$

We have to estimate the right-hand side of (4.4) term by term.

First, by Hölder's inequality and Lemma 4.3,

$$\begin{aligned}
|b(v_m, v_m, A^{1/2} v_m)| & \leq \int_\Omega |v_m| \; |\nabla v_m| \; |A^{1/2} v_m| \, dx \\
& \leq |v_m|_4 |\nabla v_m|_2 |A^{1/2} v_m|_4 \\
& \leq C |A^{1/4} v_m|_2 |A^{1/2} v_m|_2 |A^{3/4} v_m|_2 \qquad (4.5) \\
& \leq \frac{\nu}{8} |A^{3/4} v_m|_2^2 + \frac{C}{\nu} |A^{1/4} v_m|_2^2 |A^{1/2} v_m|_2^2.
\end{aligned}$$

Next, using (3.2) and Cauchy inequality,

$$\begin{aligned}
|b(\psi, v_m, A^{1/2} v_m)| & \leq \int_\Omega |\psi| \; |\nabla v_m| \; |A^{1/2} v_m| \, dx \\
& \leq C \|\varphi\|_{L^\infty(\partial\Omega)} |A^{1/2} v_m|_2^2 \qquad (4.6) \\
& \leq C \|\varphi\|_{L^\infty(\partial\Omega)} |A^{1/4} v_m|_2 |A^{3/4} v_m|_2 \\
& \leq \frac{\nu}{8} |A^{3/4} v_m|_2^2 + \frac{C}{\nu} \|\varphi\|_{L^\infty(\partial\Omega)}^2 |A^{1/4} v_m|_2^2.
\end{aligned}$$

Similarly, by (3.2) and (3.3),

$$|b(v_m, \psi, A^{1/2} v_m)| \leq \int_\Omega |v_m| \; |\nabla \psi| \; |A^{1/2} v_m| \, dx$$

$$
\begin{aligned}
&\le C\|\varphi\|_{L^\infty(\partial\Omega)} \int_\Omega \frac{|v_m(x)|}{\mathrm{dist}(x,\partial\Omega)}\, |A^{1/2}v_m|\, dx \\
&\le C\|\varphi\|_{L^\infty(\partial\Omega)} |A^{1/2}v_m|_2^2 \qquad (4.7)\\
&\le \frac{\nu}{8}|A^{3/4}v_m|_2^2 + \frac{C}{\nu}\|\varphi\|^2_{L^\infty(\partial\Omega)} \cdot |A^{1/4}v_m|_2^2.
\end{aligned}
$$

We now estimate $|\langle \bar f, A^{1/2}v_m\rangle|$ by

$$
\begin{aligned}
&|\langle f, A^{1/2}v_m\rangle| + \nu|\langle F, A^{1/2}v_m\rangle| \\
&\le |f|_2|A^{1/2}v_m|_2 + \nu\int_\Omega |F|\ |A^{1/2}v_m|\, dx \\
&\le |f|_2 \cdot |A^{1/2}v_m|_2 + c\,\nu\sqrt{\varepsilon}\int_\Omega |F| \cdot \frac{|A^{1/2}v_m|}{\mathrm{dist}(x,\partial\Omega)^{1/2}}\, dx \\
&\le |f|_2 \cdot \frac{|A^{3/4}v_m|_2}{\lambda_1^{1/4}} + c\,\nu\sqrt{\varepsilon}\,|F|_2 \left\{\int_\Omega \frac{|A^{1/2}v_m|^2}{\mathrm{dist}(x,\partial\Omega)}\, dx\right\}^{1/2} \qquad (4.8)\\
&\le |A^{3/4}v_m|_2 \left\{\frac{|f|_2}{\lambda_1^{1/4}} + c\,\nu\sqrt{\varepsilon}\,|F|_2\right\} \\
&\le \frac{\nu}{8}|A^{3/4}v_m|_2^2 + \frac{C}{\nu}\left\{\frac{|f|_2}{\lambda_1^{1/4}} + \nu\sqrt{\varepsilon}\,|F|_2\right\}^2
\end{aligned}
$$

where we used the fact $\mathrm{supp}\, F \subset \{x \in \Omega;\ c_1\,\varepsilon \le \mathrm{dist}(x,\partial\Omega) \le c_2\,\varepsilon\}$ in the second inequality and Lemma 4.1 in fourth inequality.

Finally, we estimate $|b(\psi,\psi,A^{1/2}v_m)|$. By Propositions 2.13 and 2.14,

$$
\begin{aligned}
|b(\psi,\psi,A^{1/2}v_m| &\le \int_\Omega |\psi|\ |\nabla\psi|\ A^{1/2}v_m|\, dx \\
&\le C\,\|\varphi\|_{L^\infty(\partial\Omega)} \left\{\int_\Omega |\nabla\psi|^2\,\mathrm{dist}(x,\partial\Omega)\, dx\right\}^{1/2} \\
&\qquad \times \left\{\int_\Omega |A^{1/2}v_m|^2\, \frac{dx}{\mathrm{dist}(x,\partial\Omega)}\right\}^{1/2} \qquad (4.9)\\
&\le C\|\varphi\|_{L^\infty(\partial\Omega)}\|\varphi\|_{L^2(\partial\Omega)}|A^{3/4}v_m|_2 \\
&\le \frac{\nu}{8}|A^{3/4}v_m|_2^2 + \frac{C}{\nu}\|\varphi\|^2_{L^\infty(\partial\Omega)}\,\|\varphi\|^2_{L^2(\partial\Omega)}.
\end{aligned}
$$

Putting (4.4)-(4.9) together, we obtain

$$
\begin{aligned}
\frac{d}{dt}|A^{1/4}v_m|_2^2 &\le |A^{1/4}v_m|_2^2\left\{\frac{C}{\nu}|A^{1/2}v_m|_2^2 + \frac{C}{\nu}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right\} \\
&\quad + \frac{C}{\nu}\left\{\frac{|f|_2}{\lambda_1^{1/4}} + C\nu\sqrt{\varepsilon}\,|F|_2\right\}^2 + \frac{C}{\nu}\|\varphi\|^2_{L^\infty(\partial\Omega)}\|\varphi\|^2_{L^2(\partial\Omega)}.
\end{aligned}
$$

It then follows from Gronwall's inequality that

$$
\begin{aligned}
|A^{1/4}v_m(t)|_2^2 \;\leq\; & \exp\left\{ \int_s^t \{\frac{C}{\nu}\|\varphi\|_{L^\infty(\partial\Omega)}^2 + \frac{C}{\nu}|A^{1/2}v_m|_2^2\}\, d\tau \right\} \\
& \times\left\{ |A^{1/4}v_m(s)|_2^2 + \frac{C}{\nu}\left[\frac{|f|_2}{\lambda_1^{1/4}} + C\nu\sqrt{\varepsilon}|F|_2\right]^2 (t-s) \right. \\
& \left. + \frac{C}{\nu}\|\varphi\|_{L^\infty(\partial\Omega)}^2\|\varphi\|_{L^2(\partial\Omega)}^2 (t-s) \right\}.
\end{aligned} \tag{4.10}
$$

We are now in a position to prove

Theorem 4.11 *Let $f,\ v_0 \in \mathcal{H}$ and let $v(t)$ be the unique solution of (3.1) given in Theorem 3.13. Suppose $|v_0|_2 \leq M$. Then there exists a constant $C = C(\nu, \Omega, \varphi, f, M)$ such that*

$$\sup_{t \geq \frac{1}{\nu\lambda_1}} |A^{1/4}v(t)|_2 \leq C.$$

Proof. It follows from (3.11)-(3.12) that

$$
\begin{aligned}
& \nu \int_s^t |A^{1/2}v_m(\tau)|_2^2\, d\tau \\
& \quad \leq |v_0|_2^2 + \frac{C}{\nu}\left\{\frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}\|\varphi\|_{L^\infty(\partial\Omega)}^2\right\}^2 \left\{t - s + \frac{1}{\nu\lambda_1}\right\}.
\end{aligned}
$$

Let $t - s = 1/\nu\lambda_1$. Then

$$
\begin{aligned}
& |\{\tau \in [s,t];\ |A^{1/2}v_m(\tau)|_2 > \rho\}| \\
& \quad \leq \frac{1}{\nu\rho^2}\left\{|v_0|_2^2 + \frac{C}{\nu^2\lambda_1}\left[\frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}\|\varphi\|_{L^\infty(\partial\Omega)}^2\right]^2\right\}.
\end{aligned}
$$

This implies that, if we choose

$$\rho^2 = 2\lambda_1\left\{|v_0|_2^2 + \frac{C}{\nu^2\lambda_1}\left[\frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}\|\varphi\|_{L^\infty(\partial\Omega)}^2\right]\right\},$$

then

$$|\{\tau \in [s,t];\ \ |A^{1/2}v_m(\tau)|_2 > \rho\}| \leq \frac{1}{2\lambda_1\nu}.$$

It follows that, in any interval of length $1/(\nu\lambda_1)$, there exists τ such that

$$
\begin{aligned}
&|A^{1/2}v_m(\tau)|_2^2 \le \rho^2 \\
&= 2\lambda_1 \left\{ |v_0|_2^2 + \frac{C}{\nu^2\lambda_1}\left[\frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right]^2\right\}.
\end{aligned}
$$

This, together with (4.10), (3.12) and $|A^{1/4}v_m(\tau)|_2 \le |A^{1/2}v_m(\tau)|_2/\lambda_1^{1/4}$, gives

$$
\begin{aligned}
&\sup_{t\ge 1/(\nu\lambda_1)} |A^{1/4}v_m(t)|_2^2 \\
&\le \exp\left(\frac{|v_0|_2^2}{\nu^2}\right. \\
&\left.+\frac{C}{\nu^2\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{C}{\nu^4\lambda_1}\left[\frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right]^2\right) \\
&\times\left\{2\sqrt{\lambda_1}|v_0|^2 + \frac{C}{\nu^2\sqrt{\lambda_1}}\left[\frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}|\varphi|^2_{L^\infty(\partial\Omega)}\right]^2\right. \\
&\quad\left.+\frac{C}{\nu^2\lambda_1}\left[\frac{|f|_2}{\lambda_1^{1/4}} + \frac{\nu}{\varepsilon}\|\varphi\|_{L^2(\partial\Omega)}\right]^2 + \frac{C}{\nu^2\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}\|\varphi\|^2_{L^2(\partial\Omega)}\right\}
\end{aligned}
$$

where we also used $|F|_2 \le C\|\varphi\|_{L^2(\partial\Omega)}/\varepsilon^{3/2}$ (Proposition 2.16). Thus we have shown that, if $|v_0|_2 \le M$, then

$$
\sup_{t\,\ge\,1/(\nu\lambda_1)} |A^{1/4}v_m(t)|_2 \le C(\nu,\Omega,\varphi,f,M).
$$

Now, suppose $v_{m_j} \to v$ weakly in $L^2((0,T);V)$. Since $V \hookrightarrow \mathcal{D}(A^{1/4})$ is compact, we conclude that there exists a subsequence, still denoted by $\{v_{m_j}\}$, such that $v_{m_j} \to v$ in $L^2((0,T);\mathcal{D}(A^{1/4}))$ (see [5, Lemma 8.4]). Thus there exists another subsequence, still denoted by $\{v_{m_j}\}$, so that $v_{m_j}(t) \to v(t)$ in $\mathcal{D}(A^{1/4})$ for a.e. t. It follows that

$$
\sup_{t\,\ge\,1/(\nu\lambda_1)} |A^{1/4}v(t)|_2 \le c(\nu,\Omega,\varphi,f,M).
$$

∎

Remark. For any $t_0 \in (0, \frac{1}{\nu\lambda_1})$, one may estimate

$$
\sup_{t_0 \le t \le \frac{1}{\nu\lambda_1}} |A^{1/4}v(t)|_2
$$

by integrating (4.10) with respect to s over $[t_0, t]$. We omit the details.

5 The Existence and Dimension of the Universal Attractor

Let $v(t) = S(t)v_0$ denote the solution of (3.1). We say that $\mathcal{A} \subset \mathcal{H}$ is a universal attractor for the semigroup $\{ S(t) \}_{t \geq 0}$ if $\mathcal{A}$ is a compact invariant set ($S(t)\mathcal{A} = \mathcal{A}$) which attracts the bounded sets of $\mathcal{H}$.

Theorem 5.1 *The semigroup $S(t) : \mathcal{H} \to \mathcal{H}$ possesses a universal attractor $\mathcal{A}$.*

Proof. To show that $S(t)$ has a universal attractor, it suffices to find a compact set $\mathcal{B}$ which absorbs bounded sets of $\mathcal{H}$. Then the universal attractor is given by

$$\mathcal{A} = \bigcap_{t \geq 0} S(t)\mathcal{B}.$$

(see [16, Chapter 1]).

Let $\mathcal{B} = \{u \in \mathcal{D}(A^{1/4}); |A^{1/4}u|_2 \leq \rho\}$ where $\rho > 0$ is to be determined later. Clearly $\mathcal{B}$ is compact in $\mathcal{H}$. Let $v_0 \in \mathcal{H}$ and $|v_0|_2 \leq M$. By (3.11) and a limiting argument,

$$|S(t)v_0|_2 \leq \frac{C}{\nu\sqrt{\lambda_1}} \left\{ \frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}} \|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\, |\partial\Omega|^{1/2} \|\varphi\|^2_{L^\infty(\partial\Omega)} \right\} = N$$

if $t \geq t_0(\nu, \Omega, \varphi, f, M))$. It then follows from Theorem 4.11 that, if $t \geq t_0 + \dfrac{1}{\nu\lambda_1}$, then

$$|A^{1/4}S(t)v_0|_2 \leq C(\nu, \Omega, \varphi, f, N).$$

Finally, let $\rho = C(\nu, \Omega, \varphi, f, N)$. We see that $S(t)v_0 \in \mathcal{B}$ if $t \geq t_0 + \dfrac{1}{\nu\lambda_1}$. Hence, $\mathcal{B}$ absorbs any bounded set of $\mathcal{H}$. ∎

To estimate the dimension of the universal attractor $\mathcal{A}$, we shall apply the abstract machinery in [16, Chapter 5]. To this end, we first need to show that $S(t)$ is uniformly differentiable on bounded subsets of $\mathcal{H}$.

Let

$$Rv = B(\psi, v) + B(v, \psi). \tag{5.2}$$

Then the first variation equation of (3.1) can be written in the form

$$\begin{cases} \dfrac{dU}{dt} + \nu AU + RU + B(S(t)v_0, U) + B(U, S(t)v_0) = 0 \\ U(0) = \xi \in \mathcal{H}. \end{cases} \tag{5.3}$$

Note that, by (3.6)-(3.8).

$$|\langle Rv, v\rangle| \le \frac{\nu}{4}\|v\|^2 \ \text{ for } v \in V. \tag{5.4}$$

Using (5.4) and the standard energy estimates, we may show that the linear equation (5.3) has a unique solution $U \in L^2((0,T);V) \cap C([0,T],\mathcal{H})$ for any $T > 0$. For each $t \ge 0$, $v_0 \in \mathcal{H}$, we define the linear operator $L(t, v_0) : \ \mathcal{H} \to \mathcal{H}$ by

$$L(t, v_0) \cdot \xi = U(t) \tag{5.5}$$

where $U(t)$ is the solution of (5.3).

Theorem 5.6 *Let X be a bounded subset of $\mathcal{H}$. Then*

(i) $\sup_{v_0 \in X} \|L(t, v_0)\|_{\mathcal{L}(H)} \le C\exp(c_\nu \sup_{v_0 \in X} \int_0^t \|S(\tau)v_0\|^2 \, d\tau)$,

(ii) for any $\delta > 0$,

$$\sup_{\substack{u_0, v_0 \in X \\ 0 < |u_0 - v_0| < \delta}} \frac{|S(t)v_0 - S(t)u_0 - L(t, u_0) \cdot (v_0 - u_0)|_2}{|v_0 - u_0|_2} \le C_\nu \delta \exp\left\{c_\nu \sup_{u_0 \in X} \int_0^t \|s(\tau)u_0\|^2 \, d\tau\right\}.$$

In particular, $S(t)$ is uniformly differentiable on X.

We remark that, with (5.4), Theorem 5.6 follows from the usual energy estimates exactly as in the classical case $\varphi = 0$ (see [16, Section 6.8]).

Our next result gives the estimate for the dimension of the attractor. Recall that G, the Grashof number and Re, the Reynolds number are defined in (1.10).

Theorem 5.7 *The Hausdorff and fractal dimensions of the universal attractor $\mathcal{A}$ for the semigroup $S(t)$ are bounded by*

$$c_1 G + c_2 \, Re^{3/2} + 1$$

where c_1 and c_2 are scale invariant constants depending on Ω.

Proof. With Theorems 5.1 and 5.6 at our disposal we may apply the abstract framework in [16, Chapter 5].

For $\xi_1, \xi_2, \cdots, \xi_m \in \mathcal{H}$, let $U_j(t) = L(t, v_0) \cdot \xi_j$ where $v_0 \in \mathcal{H}$. Let $Q_m(\tau)$ denote the projector from $\mathcal{H}$ to $\text{span}\{U_j(\tau) : j = 1, 2, \ldots, m\}$. Then

$$\|U_1(t) \wedge \cdots \wedge U_m(t)\|_{\Lambda^m(H)} = \|\xi_1 \wedge \ldots \wedge \xi_m\|_{\Lambda^m(H)} \exp \int_0^t \text{Tr}\, F'(S(\tau)v_0) \circ Q_m(\tau)\, d\tau$$

where $F'(S(\tau)v_0)$ is the Fréchet differential of the operator $F = -\nu A - R - B(\cdot, \cdot) + \overline{f} - B(\psi, \psi)$ at $S(\tau)v_0$:

$$F'(S(\tau)v_0) = -\nu A - R - B(S(\tau)v_0, \cdot) - B(\cdot, S(\tau)v_0). \tag{5.8}$$

Let $\{\varphi_j(\tau); \quad j = 1, 2, \ldots, m\}$ be an orthonormal basis for $\text{span}\{U_j(\tau); \quad j = 1, 2, \ldots, m\}$. Since $U_j \in L^2(0, T; V)$, $U_j(\tau) \in V$ for a.e. τ. Hence $\varphi_j(\tau) \in V$ for a.e. τ.

Note that

$$\begin{aligned}
&\text{Tr}\, F'(S(\tau)v_0) \circ Q_m(\tau) \\
&= \sum_{j=1}^m \langle F'(S(\tau)v_0)\varphi_j(\tau), \varphi_j(\tau)\rangle \\
&= \sum_{j=1}^m \Big\{ -\nu\|\varphi_j(\tau)\|^2 - \langle R\varphi_j(\tau), \varphi_j(\tau)\rangle \\
&\qquad -b(S(\tau)v_0, \varphi_j(\tau), \varphi_j(\tau)) - b(\varphi_j(\tau), S(\tau)v_0, \varphi_j(\tau)\Big\} \\
&\le -\frac{3\nu}{4} \sum_{j=1}^m \|\varphi_j(\tau)\|^2 + \sum_{j=1}^m |b(\varphi_j(\tau), S(\tau)v_0, \varphi_j(\tau))|
\end{aligned}$$

where we used (5.4) in the inequality. The second term above is bounded by

$$\int_\Omega \sum_{j=1}^m |\varphi_j(\tau)|^2 |\nabla S(\tau)v_0|\, dx \le \|S(\tau)v_0\|\ |\rho(\tau, \cdot)|_2$$

where

$$\rho(\tau, x) = \sum_{j=1}^m |\varphi_j(\tau, x)|^2.$$

By the vector valued Lieb-Thirring inequality,

$$|\rho(\tau,\cdot)|_2^2 \le C \sum_{j=1}^{m} \|\varphi_j(\tau)\|^2.$$

It follows that

$$\begin{aligned}
\operatorname{Tr} F'(S(\tau)v_0)\circ Q_m(\tau) &\le -\frac{3\nu}{4}\sum_{j=1}^{m}\|\varphi_j(\tau)\|^2 + C\|S(\tau)v_0\|\left(\sum_{j=1}^{m}\|\varphi_j(\tau)\|^2\right)^{1/2}\\
&\le -\frac{\nu}{2}\sum_{j=1}^{m}\|\varphi_j(\tau)\|^2 + \frac{C}{\nu}\|S(\tau)v_0\|^2 \le -\frac{\nu}{2}\sum_{j=1}^{m}\lambda_j + \frac{C}{\nu}\|S(\tau)v_0\|^2\\
&\le -\frac{\pi\nu}{2|\Omega|}\, m^2 + \frac{C}{\nu}\|S(\tau)v_0\|^2
\end{aligned}$$

where we have used the variational principle in the third inequality and $\sum_{j=1}^{m}\lambda_j \ge \pi m^2/|\Omega|$ in the fourth (see [8]).

Now, let

$$q_m(t) = \sup_{v_0\,\in\,\mathcal{A}} \ \sup_{\substack{\xi_j\,\in\,\mathcal{H}\\ j\,=\,1,2,\cdots,m}} \left\{\frac{1}{t}\int_0^t \operatorname{Tr} F'(S(\tau)v_0)\circ Q_m(\tau)\,d\tau\right\}.$$

Then

$$q_m(t) \le -\frac{\pi\nu}{2|\Omega|}m^2 + \frac{C}{\nu}\sup_{v_0\,\in\,\mathcal{A}}\frac{1}{t}\int_0^t \|S(\tau)v_0\|^2\,d\tau.$$

Hence,

$$q_m \equiv \limsup_{t\,\to\,\infty}\ q_m(t) \le -\frac{\pi\nu}{2|\Omega|}m^2 + \frac{C}{\nu}\cdot\gamma \tag{5.9}$$

where

$$\gamma = \limsup_{t\,\to\,\infty}\ \sup_{v_0\,\in\,\mathcal{A}}\ \frac{1}{t}\int_0^t \|S(\tau)v_0\|^2\,d\tau. \tag{5.10}$$

It follows from (5.9) and the general result in [16, Chapter 4] that the Hausdorff and fractal dimensions of the universal attractor are $\mathcal{A}$ bounded by

$$\frac{C|\Omega|^{1/2}}{\nu}\gamma^{1/2} + 1.$$

It remains to estimate γ defined by (5.10).

By (3.12) and a limiting argument,

$$\nu \int_0^t \|S(\tau)v_0\|^2\, d\tau \le |v_0|_2^2 + \frac{C\,t}{\nu}\left\{ \frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right\}^2$$

where ε is defined in (3.7). It follows that

$$\gamma \le \frac{C}{\nu^2}\left\{ \frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right\}^2.$$

Hence,

$$\dim\mathcal{A} \le \frac{C|\Omega|^{1/2}}{\nu^2}\left\{ \frac{|f|_2}{\sqrt{\lambda_1}} + \frac{\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\,|\partial\Omega|^{1/2}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right\} + 1.$$

Since $\lambda_1 \sim 1/|\Omega|$, $|\partial\Omega| \sim |\Omega|^{1/2}$ and $\|\varphi\|_{L^2(\partial\Omega)} \le |\partial\Omega|^{1/2}\|\varphi\|_{L^\infty(\partial\Omega)}$, we have

$$\dim\mathcal{A} \le \frac{C|f|_2}{\nu^2\lambda_1} + \frac{1}{\sqrt{\varepsilon}}\cdot\frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu\lambda_1^{3/4}} + \sqrt{\varepsilon}\cdot\frac{C\|\varphi\|^2_{L^\infty(\partial\Omega)}}{\nu^2\lambda_1^{3/4}} + 1$$

where the constants depend on the scale-invariant quantities $|\Omega|^{1/2}/|\partial\Omega|$ and $(|\Omega|\lambda_1)^{-1}$.

Finally, by (3.7),

$$\sqrt{\varepsilon} \le \frac{C\sqrt{\nu}}{\|\varphi\|^{1/2}_{L^\infty(\partial\Omega)}} \quad\text{and}\quad \frac{1}{\sqrt{\varepsilon}} \le C\left\{\frac{\|\varphi\|^{1/2}_{L^\infty(\partial\Omega)}}{\sqrt{\nu}} + \lambda_1^{1/4}\right\}.$$

We obtain

$$\begin{aligned}\dim\mathcal{A} &\le \frac{C|f|_2}{\nu^2\lambda_1} + \frac{C\|\varphi\|^{3/2}_{L^\infty(\partial\Omega)}}{\nu^{3/2}\lambda_1^{3/4}} + \frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu\lambda_1^{1/2}} + 1\\ &= C\,G + C\ \mathrm{Re}^{3/2} + C\ \mathrm{Re} + 1\\ &\le c_1 G + c_2\ \mathrm{Re}^{3/2} + 1\end{aligned}$$

where $G = \dfrac{|f|_2}{\nu^2\lambda_1}$ and $\mathrm{Re} = \dfrac{\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu\lambda_1^{1/2}}$.

The proof of the theorem is now finished. ∎

A Appendix: The Stokes System in Two-dimensional Lipschitz Domains

In this appendix, we sketch the proof of Theorem 2.3. We will only indicate the modifications which are needed to carry over the arguments of Fabes, Kenig and Verchota [6] and Shen [12] to the two-dimensional case.

Let $\Gamma(x) = (\Gamma_{jk}(x))_{1 \le j,k \le 2}$ be a matrix of fundamental solutions and $P(x) = (P_1(x), P_2(x))$ the corresponding pressure vector for the Stokes system in $\mathbf{R}^2$ where

$$\begin{cases} \Gamma_{jk}(x) = \dfrac{1}{4\pi}\left\{-\delta_{jk}\log|x| + \dfrac{x_j x_k}{|x|^2}\right\} \\ P_i(x) = \dfrac{1}{2\pi}\cdot\dfrac{x_i}{|x|}. \end{cases} \tag{A.1}$$

Following [6], we use the method of layer potentials to solve the L^2-Dirichlet problem. The complication for the two-dimensional case comes from the fact that $|\Gamma(x)| \to \infty$ as $|x| \to \infty$. The problem can be solved easily by first restricting the density function to the subspace

$$L_0^2(\partial\Omega) = \left\{ f \in L^2(\partial\Omega); \quad \int_{\partial\Omega} f\, d\sigma = 0 \right\}.$$

Given $f \in L^2(\partial\Omega)$, we define the single layer potential

$$\mathcal{S}(f)(x) = \int_{\partial\Omega} \Gamma(x-Q) f(Q)\, d\sigma(Q) \tag{A.2}$$

and the corresponding pressure

$$q(x) = \int_{\partial\Omega} P(x-Q)\cdot f(Q)\, d\sigma(Q). \tag{A.3}$$

Consider the conormal derivative on $\partial\Omega$

$$\frac{\partial u}{\partial \rho(Q)} = \frac{\partial u}{\partial n(Q)} - q(Q) n(Q) \tag{A.4}$$

where $n(Q)$ always denotes the outward unit normal to $\partial\Omega$ at Q.

Let $\Omega_+ = \Omega$ and $\Omega_- = (\overline{\Omega})^c$. If $u = \mathcal{S}(f)$, then

$$\frac{\partial u_\pm}{\partial \rho(Q)} = \left(\pm\frac{1}{2}I + K\right) f(Q) \tag{A.5}$$

where $\pm$ indicate the nontangential limits taken from $\Omega_\pm$ respectively, and K is a bounded singular integral operator on $L^p(\partial\Omega)$, $1 < p < \infty$.

For $f \in L^2(\partial\Omega)$, we define the double-layer potential

$$u(x) = \mathcal{K}f(x) = \int_{\partial\Omega} \frac{\partial}{\partial\rho(Q)}\{\Gamma(x-Q)\} f(Q)\, d\sigma(Q). \tag{A.6}$$

Then

$$u_\pm(Q) = \left(\mp\frac{1}{2}I + K^*\right) f(Q) \tag{A.7}$$

where K^* is the adjoint operator of K in (A.5).

Let R denote the orthogonal complement to the kernel of $-\frac{1}{2}I + K$. To show the existence of solutions to the L^2-Dirichlet problem, it suffices to prove that

$$-\frac{1}{2}I + K^* : \quad R \to L^2_n(\partial\Omega) = \left\{ f \in L^2(\partial\Omega); \quad \int_{\partial\Omega} f \cdot n \, d\sigma = 0 \right\}$$

is invertible. By duality, it is enough to show that $-\frac{1}{2}I + K$ is invertible from $L^2_n(\partial\Omega)$ to a subspace of $L^2(\partial\Omega)$ of codimension one.

Proposition A.8 *The operator* $-\frac{1}{2}I + K : \ L^2_n(\partial\Omega) \to L^2(\partial\Omega)$ *is one-to-one.*

Proof. Suppose $f \in L^2_n(\partial\Omega)$ and $\left(-\frac{1}{2}I + K\right) f = 0$. Let $u = \mathcal{S}(f)$ be the single layer potential defined by (A.2). Then

$$\frac{\partial u_-}{\partial \rho} = \left(-\frac{1}{2}I + K\right) f = 0 \quad \text{a.e. on } \partial\Omega.$$

Since $f = \frac{\partial u_+}{\partial \rho} - \frac{\partial u_-}{\partial \rho}$ and

$$\int_{\partial\Omega} \frac{\partial u_+}{\partial \rho} \, d\sigma = 0,$$

we obtain

$$\int_{\partial\Omega} f \, d\sigma = 0.$$

This implies that

$$u(x) = \int_{\partial\Omega} \{\Gamma(x - Q) - \Gamma(x)\} f(Q) \, d\sigma(Q) \to 0 \quad \text{as } |x| \to \infty. \tag{A.9}$$

It then follows from the divergence theorem and a limiting argument that

$$\int_{\Omega_-} |\nabla u(x)|^2 \, dx = -\int_{\partial\Omega} \frac{\partial u_-}{\partial \rho} u \, d\sigma = 0.$$

Hence $u =$ constant in Ω_-. But $u(x) \to 0$ as $|x| \to \infty$, thus, $u \equiv 0$ in Ω_-. The rest of the proof is the same as in [6, Lemma 2.1] ∎

Proposition A.10 *The operator* $-\frac{1}{2}I+K: \quad L^2(\partial\Omega)\to L^2(\partial\Omega)$ *has a closed range.*

Proof. It is not hard to check that the formulas (1.2)-(1.7) in [6] hold when $d=2$ except

$$-\int_{\Omega_-}|\nabla u|^2=\int_{\partial\Omega}\frac{\partial u_-}{\partial\rho}u\,d\sigma, \tag{A.11}$$

if $u=\mathcal{S}(f)$ and $f\in L^2(\partial\Omega)$. As we see in the proof of Proposition A.8, (A.11) holds if we assume $f\in L^2_0(\partial\Omega)$. Thus, by Lemma 1.17 in [6], if $f\in L^2_0(\partial\Omega)$,

$$\begin{aligned}\|f\|_{L^2(\partial\Omega)}&\le\left\|\frac{\partial u_+}{\partial\rho}\right\|_{L^2(\partial\Omega)}+\left\|\frac{\partial u_-}{\partial\rho}\right\|_{L^2(\partial\Omega)}\le C\left\{\left\|\frac{\partial u_-}{\partial\rho}\right\|_{L^2(\partial\Omega)}+\left|\int_{\partial\Omega}u\,d\sigma\right|+|q(0)|\right\}\\&=C\left\{\left\|\left(-\frac{1}{2}I+K\right)f\right\|_{L^2(\partial\Omega)}+\left|\int_{\partial\Omega}u\,d\sigma\right|+|q(0)|\right\}\end{aligned}$$

where we have assumed $0\in\Omega$. This, together with the fact that $L^2_0(\partial\Omega)$ is a subspace of $L^2(\partial\Omega)$ of codimension two, implies that the range of $-\frac{1}{2}I+K$ is closed, by a rather standard argument. ∎

We are now ready to give the

Proof of Theorem 2.3. Given $\varphi\in L^2_n(\partial\Omega)$, the existence and uniqueness of the solution (u,q) satisfying (2.2) and $(u)^*\in L^2(\partial\Omega)$ follow from Propositions A.8 , A.10 and an approximation argument as in [6]. We remark that the uniqueness also follows from the estimate

$$\int_{\Omega_j}|u|^2\,dx\le C\int_{\partial\Omega_j}|u|^2\,d\sigma \tag{A.12}$$

where $\{\Omega_j\}$ is a sequence of smooth domains approximating Ω. (A.12) can be established by using Rellich identities in a manner similar to the proof of Lemma 5.1.14 in [11].

The proof of the square function estimates,

$$\int_\Omega|\nabla u(x)|^2\operatorname{dist}(x,\partial\Omega)\,dx+\int_\Omega|q(x)|^2\operatorname{dist}(x,\partial\Omega)dx\le C\int_{\partial\Omega}|u|^2\,d\sigma,$$

is the same as in the higher dimensional case (see e.g. [2]).

Finally we show

$$\sup_{x \in \Omega} |u(x)| \le C\|\varphi\|_{L^\infty(\partial\Omega)}. \tag{A.13}$$

The estimate

$$\sup_{x \in \Omega} |\nabla u(x)| \operatorname{dist}(x, \partial\Omega) \le C\|\varphi\|_{L^\infty(\partial\Omega)}$$

follows easily from (A.13) and the standard interior estimates.

Since (A.13) is dilation invariant, we may assume $\operatorname{diam}\Omega = 1$. Given $z \in \Omega$, we wish to show that

$$|u(z)| \le C\|\varphi\|_{L^\infty(\partial\Omega)}. \tag{A.14}$$

Let $r = \operatorname{dist}(z, \partial\Omega)$. We introduce another matrix of fundamental solutions $\widetilde{\Gamma}(x) = (\widetilde{\Gamma}_{jk}(x))_{1\le j,k\le 2}$ where

$$\widetilde{\Gamma}_{jk}(x) = \frac{1}{4\pi}\left\{-\delta_{jk}\log\left(\frac{|x|}{r}\right) + \frac{x_j x_k}{|x|^2}\right\}. \tag{A.15}$$

We construct the matrix Green's function $G(x, y)$ and the corresponding pressure vector $(\pi^x(y))$ where

$$\begin{cases} G(x,y) = \widetilde{\Gamma}(x-y) - v^x(y) \\ \\ \pi^x(y) = P(x-y) - q^x(y), \end{cases} \tag{A.16}$$

and, for each $x \in \Omega$, $(v^x(y), q^x(y))$ is the matrix-valued solution to the L^2 Dirichlet problem (2.2) with boundary data $v^x(Q) = \widetilde{\Gamma}(x - Q)$ on $\partial\Omega$.

Since

$$u(x) = \int_{\partial\Omega} \frac{\partial G}{\partial\rho(Q)}(x, Q)\varphi(Q)\, d\sigma(Q), \quad \text{for } x \in \Omega,$$

and $\varphi \in L^2_n(\partial\Omega)$, (A.14) follows from

$$\int_{\partial\Omega} \left|\frac{\partial G}{\partial\rho(Q)}(z, Q) - \pi^z(x_0)n(Q)\right| d\sigma(Q) \le C \tag{A.17}$$

for some $x_0 \in \Omega$.

Let $Q_0 \in \partial\Omega$ such that $|z - Q_0| = \operatorname{dist}(z, \partial\Omega) = r$. The proof of

$$\int_{\substack{|Q-Q_0|\le 30r \\ Q\in\partial\Omega}} \left|\frac{\partial G}{\partial\rho(Q)}\,(z, Q) - \pi^z(x_0)n(Q)\right| d\sigma(Q) \le C \tag{A.18}$$

is exactly the same as in [12, p.807].

By the proof of Lemma 1.7 in [12], we have

$$\int_{\substack{R \leq |Q-Q_0| \leq 2R \\ Q \in \partial\Omega}} \left| \frac{\partial G}{\partial \rho(Q)}(z,Q) - \pi^z(x_0) n(Q) \right| d\sigma(Q) \leq C \left(\frac{r}{R}\right)^{1/2} \tag{A.19}$$

for $R \geq 30r$ if we can show

$$\int_{\substack{|Q-Q_0| \geq 10r \\ Q \in \partial\Omega}} |(G(z,\cdot))^*(Q)|^2 d\sigma(Q) \leq C\, r. \tag{A.20}$$

(A.17) follows easily from (A.18)-(A.19) by summation.

To see (A.20), we apply the L^2-estimate on the domain $\Omega \setminus B(Q_0, 4r) = \{x \in \Omega;\ |x - Q_0| > 4r\}$. We obtain

$$\int_{\substack{|Q-Q_0| \geq 10r \\ Q \in \partial\Omega}} |(G(z,\cdot))^*(Q)|^2 d\sigma(Q) \leq C \int_{\Omega \cap \partial B(Q_0,4r)} |G(z,Q)|^2\, d\sigma(Q) \tag{A.21}$$

$$\leq C \int_{\Omega \cap \partial B(Q_0,4r)} |\widetilde{\Gamma}(z-Q)|^2\, d\sigma(Q) + C \int_{\Omega \cap \partial B(Q_0,4r)} |v^z(Q)|^2\, d\sigma(Q)$$

since $G(z,\cdot) = 0$ on $\partial\Omega$. By (A.16),

$$\int_{\Omega \cap \partial B(Q_0,4r)} |\widetilde{\Gamma}(z-Q)|^2\, d\sigma(Q) \leq C\, r.$$

Using $|v^z(Q)| = |\widetilde{\Gamma}(z-Q)| \leq C$ for $Q \in \partial\Omega \cap \partial B(Q_0, 4r)$, we get

$$\begin{aligned}
\int_{\Omega \cap \partial B(Q_0,4r)} |v^z(Q)|^2\, d\sigma(Q) &\leq C\, r + C\, r^2 \int_{\Omega \cap \partial B(Q_0,4r)} |\nabla_Q v^z(Q)|^2 d\sigma(Q) \\
&\leq C\, r + C\, r^2 \int_{\partial\Omega} |(\nabla v^z)^*|^2\, d\sigma \\
&\leq C\, r + C\, r^2 \int_{\partial\Omega} |\nabla v^z|^2\, d\sigma \\
&\leq C\, r + C\, r^2 \int_{\partial\Omega} |\nabla_{\tan} v^z|^2\, d\sigma
\end{aligned}$$

where $\nabla_{\tan} v^z$ denotes the tangential derivative of v^z on $\partial\Omega$ and the last inequality follows from Lemma 1.10 (i) and Lemma 1.16 (i) in [6].

Since $v^z(Q) = \widetilde{\Gamma}(z-Q)$ on $\partial\Omega$, we conclude that

$$\int_{\substack{|Q-Q_0|\geq 10r \\ Q\in\partial\Omega}} |(G(z,\cdot))^*(Q)|^2\, d\sigma(Q) \leq C\,r + C\,r^2 \int_{\partial\Omega} |\nabla_{\tan} v^z|^2\, d\sigma(Q)$$
$$\leq C\,r + C\,r^2 \int_{\partial\Omega} |\nabla_Q \widetilde{\Gamma}(z-Q)|^2\, d\sigma(Q) \leq C\,r + C\,r^2 \int_{c\,r}^{\infty} \frac{dt}{t^2} \leq C\,r.$$

(A.20) is then proved. The proof of Theorem 2.1 is complete. ∎

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