Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result

Russell M. Brown*
Department of Mathematics
University of Kentucky
Lexington, KY 40506

Abstract

A formula is given for recovering the boundary values of the coefficient γ of an elliptic operator, $\mathrm{div}\gamma\nabla$, from the Dirichlet to Neumann map. The main point is that one may recover γ without any *a priori* smoothness assumptions. The formula allows one to recover the value of γ pointwise.

Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a bounded open set with Lipschitz boundary and let $\gamma : \bar{\Omega} \rightarrow \mathbf{R}$ satisfy $\lambda^{-1} \geq \gamma(x) \geq \lambda$ for some $\lambda > 0$. Let $L_\gamma = \mathrm{div}\, \gamma \nabla$ be an elliptic operator. We let $\Lambda_\gamma : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$ denote the Dirichlet to Neumann map. Thus if u solves the Dirichlet problem,

$$\begin{cases} \mathrm{div}\, \gamma\, \nabla u = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases}$$

*Research supported by the National Science foundation and the Commonwealth of Kentucky through the Kentucky NSF-EPSCoR program.

then $\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}$. The purpose of this note is to show that we can recover $\gamma|_{\partial\Omega}$ from Λ_γ under minimal smoothness hypotheses on γ.

Since we will be working with nonsmooth γ, the equation div $\gamma \nabla u = 0$ will be interpreted in the weak sense and we define $\Lambda_\gamma f$ as an element of the dual of $H^{1/2}(\partial\Omega)$ by the relation

$$\int_{\partial\Omega} \phi \Lambda_\gamma f = \int_\Omega \gamma \nabla u \cdot \nabla \Phi$$

where u is the solution of the Dirichlet problem and $\Phi \in H^1(\Omega)$ satisfies $\Phi|_{\partial\Omega} = \phi$. Note that we are abusing notation by writing the bilinear pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ as an integral.

Much is known about the recovery of $\gamma|_{\partial\Omega}$ from Λ_γ. We recall some of the previous work. In 1984, Kohn and Vogelius [4] showed that if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 - \gamma_2$ vanishes to infinite order at the boundary provided γ_1 and γ_2 are $C^\infty(\bar{\Omega})$. Their argument depended on L^2-Sobolev embedding and, as a consequence, they need to assume that $\gamma_1 - \gamma_2$ has $n/2$ "extra derivatives" in order to obtain that $\gamma_1 - \gamma_2$ vanishes at the boundary. When γ and $\partial\Omega$ are C^∞, Sylvester and Uhlmann [10] show how to recover all derivatives of γ from the pseudo-differential operator Λ_γ. By a limiting argument, they establish that if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ provided γ_1 and γ_2 are continuous. However, the limiting behavior of the operator as the domain varies is less clear and thus they do not consider relaxing the regularity assumption on the domain. Finally, G. Alessandrini [1] has obtained uniqueness at the boundary when the coefficient is in the Sobolev space $W^{1,p}$, for some $p > n$, and the domain is Lipschitz. Recently, Nachman [5] has given a constructive method for recovering γ from Λ_γ when the coefficient lies in $W^{1,p}$, $p > n$.

In this note, we will show how to recover $\gamma|_{\partial\Omega}$ when $\partial\Omega$ is Lipschitz and, roughly speaking, γ is continuous at a.e. boundary point. The precise hypothesis will be given in (H1) below. For now, we observe that our hypothesis includes γ which lie in the Sobolev space $W^{1,1}(\Omega)$, continuous γ and certain γ which are piecewise continuous (we must have some control on the boundaries of the subregions on which γ is continuous). Thus our result implies results on recovering γ described above and also recovers discontinuous γ which had not been treated earlier.

This result is of interest for several reasons. Recovering γ on the boundary is a first step in more general results where one wants to find γ in the interior from the Dirichlet to Neumann map. The result of this paper shows that the boundary identifiability is not an impediment to recovering discontinuous γ in the interior. However, the interior identifiability remains a hard problem. See [2] for recent progress on the interior problem in two dimensions. The boundary identifiabilty result also arises as in the so-called "layer-stripping" method for solving the inverse conductivity problem (see [7, 9]). Finally, the method of this paper is quite flexible since it only relies on the standard H^1-estimates for elliptic equations. Recently, R. Robertson has adapted these methods to obtain boundary identifiability results for an equation of elasticity involving additional terms which represent residual stress [6].

We begin our development by stating the hypothesis on $\partial\Omega$. We assume that $\partial\Omega$ is a Lipschitz domain in the sense that $\partial\Omega$ is locally the graph of a Lipschitz function. Thus for each $P \in \partial\Omega$, there is a coordinate system (x', x_n) on $\mathbf{R}^{n-1} \times \mathbf{R}$, isometric to the standard one, and a Lipschitz function $\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ so that for some $\rho > 0$,

$$B(P, \rho) \cap \{x_n > \phi(x')\} = \Omega \cap B(P, \rho)$$

and

$$B(P, \rho) \cap \{x_n = \phi(x')\} = \partial\Omega \cap B(P, \rho).$$

We let F denote the map $F(x', x_n) = (x', \phi(x') + x_n)$. We choose $r > 0$ and assume the coordinates are fixed so that $F(B(0, 2r)) \subset B(P, \rho)$. In the remainder of this note, we fix P and show how to recover $\gamma \circ F$ in $B(0, r) \cap \{x_n = 0\}$.

We next state our hypothesis on γ.

H1 For each $P \in \partial\Omega$, there exist a representative of γ, a coordinate system, a map F and a ball $B(0, r)$ as in the definition of Lipschitz domain so that

$$\lim_{x_n \to 0^+} \gamma(F(x', x_n)) = \gamma(F(x', 0)), \quad \text{a.e. } x', \quad |x'| < r. \tag{1}$$

The formula for γ in the theorem below only depends on γ as an element of L^∞ but altering γ on a set of measure zero may affect the truth of (H1). Thus, we emphasize that (H1) need only hold for some representative of γ in order to obtain our Theorem. In fact, the representative for which (H1) holds may vary from point to point. We recall that if γ is in $W^{1,1}(\Omega)$, then it is known that γ has a representative for which $\gamma(F(x', \cdot))$ is absolutely continuous for a.e x' and thus γ satisfies (H1). We can now state our main result.

Theorem. *Let Ω be a Lipschitz domain and let γ be a function satisfying $\lambda \le \gamma(x) \le \lambda^{-1}$. Assume the condition (H1). Then for a.e. $P \in \partial\Omega$, there exists a family of functions f_N so that*

$$\lim_{N\to\infty} \int_{\partial\Omega} \bar{f}_N \Lambda_\gamma f_N = \gamma(P).$$

Before proceeding with the proof, we observe that if u satisfies $\operatorname{div}\gamma\nabla u = 0$, then $v = u \circ F$ satisfies $\operatorname{div} A(x)\nabla v = 0$ in $\tilde{\Omega}$

$$A(x', x_n) = \gamma(F(x', x_n))(DF^{-1}(x'))^t(DF^{-1}(x')),$$

and $\tilde{\Omega} = F^{-1}(\Omega)$. Note that the map F is biLipschitz in all of $\mathbf{R}^n$.

Our next step is to describe the points P for which we can recover $\gamma(P)$. Again, we state the condition in each coordinate system.

H2 The point x', is a Lebesgue point for $A(\cdot, 0)$ and $DF^{-1}(\cdot)$ and moreover that

$$\lim_{r\to 0^+} \frac{1}{r^{n-1}} \int_{|x'-y'|<r} |A(x', 0) - A(y', 0)|^2 \, dy' = 0. \tag{2}$$

Since A and DF^{-1} are bounded, it is well known that (H2) holds for a.e. x'.

We also need some control on the rate at which γ attains its boundary values. To state this condition, we define auxiliary functions for λ small and positive by

$$\gamma^*_\lambda(x') = \sup_{0<x_n<\lambda} |\gamma(F(x', 0)) - \gamma(F(x', x_n))|, \quad |x'| \le 3r/2.$$

The assumption (H1) on γ implies that γ^*_λ decreases to 0 a.e. as $\lambda \to 0^+$. Since γ is bounded, the monotone convergence theorem implies $\gamma^*_\lambda \to 0$ in

$L^p(\{x' : |x'| < 3r/2\}), \quad p < \infty$. We define the Hardy-Littlewood maximal operator on $\mathbf{R}^{n-1}$ by

$$M(f)(x') = \sup_{s>0} s^{1-n} \int_{|x'-y'|<s} |f(y')| \, dy'.$$

If we extend γ_λ^* to be zero outside $B(0, 3r/2)$, then our observation that $\gamma_\lambda^* \to 0$ in every L^p-space, $p < \infty$, the L^p- mapping properties of the maximal operator [8] and the monotonicity in λ of $M(\gamma_\lambda^{*2})$ imply that for a.e. x',

$$M(\gamma_\lambda^{*2})(x') \to 0. \tag{3}$$

We let $\eta : \mathbf{R} \to [0,1]$ be a smooth function which satisfies $\eta(t) = 1, \quad |t| \leq 1/2$, and $\eta(t) = 0, \quad |t| > 1$. We choose $\alpha \in \mathbf{R}^n$ a constant vector for which $A(x', 0)\alpha \cdot \alpha = A(x', 0)e_n \cdot e_n$ and $A(x', 0)\alpha \cdot e_n = 0$. Note that these condition depend only on DF^{-1} and not the value of γ. Set $\mu = DF^{-1}(x')(i\alpha - e_n)$. Then, this vector satisfies $\mu \cdot \mu = 0$ and $\bar{\mu} \cdot \mu = |DF^{-1}(x')e_n|^2 = 2(1 + |\nabla\phi(x')|^2)$. We set

$$v_N(y) = \eta(N^{1/2}|y' - x'|)\eta(N^{1/2}y_n)e^{N(i\alpha - e_n)\cdot(y-(x',0))}. \tag{4}$$

Lemma 1. *Suppose that x', A and γ satisfy (H1), (H2) and (3). Then*

$$\int_{\tilde{\Omega}} A\nabla v_N \cdot \nabla \bar{v}_N \, dy = N^{\frac{3-n}{2}} \gamma(F(x'))(1 + |\nabla\phi(x')|^2) \int_{\mathbf{R}^{n-1}} \eta(|x'|)^2 \, dx' + o(N^{\frac{3-n}{2}}).$$

Proof. We assume that $x' = 0$. We set $\psi(y) = \eta(y_n)\eta(|y'|)$, we use the definition of A and μ and obtain

$$\begin{aligned}
&\int_{\tilde\Omega} A(y)\nabla v_N(y)\cdot\nabla \bar v_N(y)\,dy \\
&\qquad = N^2\int_{\tilde\Omega}\gamma(F(0))\mu\cdot\bar\mu\psi(N^{1/2}y)^2e^{-2Ny_n}\,dy \\
&\quad + N^2\int_{\tilde\Omega}(A(y)-A(0))(i\alpha-e_n)(-i\alpha-e_n)e^{-2Ny_n}\psi(N^{1/2}y)^2\,dy \qquad (5)\\
&\quad + N\int_{\tilde\Omega}e^{-2Ny_n}A(y)(\nabla\psi)(N^{1/2}y)\cdot(\nabla\psi)(N^{1/2}y)\,dy \\
&\quad + N^{3/2}\int_{\tilde\Omega}e^{-2Ny_n}A(y)(-e_n)\cdot(\nabla\psi)(N^{1/2}y)\psi(N^{1/2}y)\,dy.
\end{aligned}$$

It is easy to see that the first term on the right of (5) satisfies

$$\begin{aligned}
&N^2\int_{\tilde\Omega}\gamma(F(0))\mu\cdot\bar\mu\psi(N^{1/2}y)^2e^{-2Ny_n}\,dy \\
&\quad = \gamma(F(0))(1+|\nabla\phi(0)|^2)N^{\frac{3-n}{2}}\int_{\mathbf{R}^{n-1}}\eta(|y'|)^2\,dy' + O(\exp(-\tfrac12 N^{1/2})N^{\frac{3-n}{2}}) \qquad (6)
\end{aligned}$$

which gives the main term in the conclusion of the Lemma. It is also easy to show that the last two terms in (5) are $O(N^{\frac{2-n}{2}})$ and $O(N^{\frac{1-n}{2}})$, respectively, and thus each is better than the allowed error term of $o(N^{\frac{3-n}{2}})$ as $N\to\infty$. To estimate the second term, we write

$$\begin{aligned}
&N^2\left|\int_{\tilde\Omega}(A(y)-A(0))(i\alpha-e_n)\cdot(-i\alpha-e_n)\psi(N^{1/2}|y|)^2e^{-2Ny_n}\,dy\right| \\
&\quad \le \tfrac12 N|i\alpha-e_n|^2\int_{\mathbf{R}^{n-1}}|A(y',0)-A(0)|\eta(N^{1/2}|y'|)^2\,dy' \\
&\quad + N^2|i\alpha-e_n|^2\|DF^{-1}\|_\infty^2\int_{\tilde\Omega}|\gamma(F(y',0))-\gamma(F(y',y_n))| \qquad (7)\\
&\qquad\qquad \times\psi(N^{1/2}|y|)^2e^{-2Ny_n}\,dy.
\end{aligned}$$

The first term on the right of (7) is $o(N^{\frac{3-n}{2}})$ by (2). To estimate the second term on the right of (7), we choose $\lambda > 0$ and split the integral into regions where $x_n > \lambda$ and $x_n < \lambda$ giving

$$\begin{aligned}\int_{\tilde{\Omega}} |\gamma(F(y',0)) - \gamma(F(y',y_n))| \quad & \psi(N^{1/2}|y|)^2 e^{-2Ny_n}\, dy \\ & \leq CN^{\frac{-1-n}{2}}(M(\gamma_\lambda^*)(0) + 2\|\gamma\|_\infty e^{-2N\lambda}).\end{aligned}$$

Since $\lambda > 0$ and 0 satisfies (3), this gives that the second term is also $o(N^{\frac{3-n}{2}})$ as $N \to \infty$. $\Box$

Our next step is to show that the functions v_N given in (4) approximate solutions. This is done in the obvious way: We compute div $A\nabla v_N$ and show that, in the norm of $H^{-1}(\Omega)$, this is $o(N^{\frac{3-n}{4}})$. Then standard energy estimates imply that the solution w_N of

$$\begin{cases} \operatorname{div} A\nabla w_N = \operatorname{div} A\nabla v_N, & \text{in } \tilde{\Omega} \\ w_N = 0, & \text{on } \partial\tilde{\Omega} \end{cases}$$

satisfies $\int |\nabla w_N|^2 = o(N^{\frac{3-n}{2}})$.

The one novel point in this argument is that we use Hardy's inequality:

$$\int_{\tilde{\Omega}} \frac{u(x)^2}{\delta(x)}\, dx \leq C \int_{\tilde{\Omega}} |\nabla u(x)|^2\, dx, \quad u \in H_0^1(\tilde{\Omega}). \tag{8}$$

where $\delta(x)$ denotes the distance between and x and $\partial\tilde{\Omega}$. This estimate holds, at least, in biLipschitz images of Lipschitz domains (see [3][p.26]). This estimate will be used to obtain optimal estimates for elements in $H^{-1}(\Omega)$ of the form $u \to \int fu$ when f is concentrated near the boundary.

Lemma 2. *Let v_N be as defined in (4) and let w_N solve*

$$\operatorname{div} A\nabla w_N = \operatorname{div} A\nabla v_N, \quad w_N \in H_0^1(\Omega),$$

then

$$\int |\nabla w_N|^2 = o(N^{\frac{3-n}{2}}).$$

Proof. Again, we assume that $x' = 0$. We will norm $H_0^1(\Omega)$ by $\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2\, dx$ and we let $H^{-1}(\Omega)$ have the standard dual norm. We recall the definition of $A(x)$ after the statement of the theorem, let $L_0 = \operatorname{div} A(0)\nabla$, set $E(x) = \exp(N(i\alpha - e_n)\cdot x)$ and $\psi(x) = \eta(N^{1/2}|x'|)\eta(N^{1/2}x_n)$ (note that this is slightly different than the ψ defined in Lemma 1). With these notations we write

$$\begin{aligned} Lv_N &= L_0(\psi E) + \operatorname{div}\,(A(x) - A(0))\nabla(\psi E) \\ &= 2A(0)\nabla\psi\cdot\nabla E + EL_0\psi \\ &\qquad + \operatorname{div}\,(A(x) - A(0))\nabla(\psi E) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

According to the Lax-Milgram lemma, we have $\|w_N\|_{H_0^1(\Omega)} \le C\|Lv_N\|_{H^{-1}(\Omega)}$. Thus we need to show

$$\|Lv_N\|_{H^{-1}(\Omega)} = o(N^{\frac{3-n}{4}}).$$

In order to carry the estimates, let ϕ be in $H_0^1(\Omega)$ and consider each of the terms I, II and III paired with ϕ. To estimate $I(\phi)$, we apply Cauchy-Schwarz, Hardy's inequality and elementary estimates for ∇E and $\nabla\psi$ to obtain

$$
\begin{aligned}
I(\phi) \quad &= 2\int A(0)\nabla\psi\cdot\nabla E\phi \\
&\le \left(\int \frac{\phi(y)^2}{\delta(y)^2}\right)^{1/2} N^{3/2}\left(\int_{|y'|<N^{-1/2}} e^{-2Ny_n}\cdot\delta(y)^2\right)^{1/2} \\
&\le C\|\phi\|_{H_0^1} N^{\frac{1-n}{4}}.
\end{aligned}
$$

The term II is also easy to estimate using Hardy's inequality and has the bound

$$
\mathrm{II}(\phi) \le C\|\phi\|_{H_0^1((0))} N^{\frac{-1-n}{4}}.
$$

Finally, for III we write

$$
\begin{aligned}
\mathrm{III}(\phi) &= -\int_{\tilde\Omega} (A(y)-A(0))\nabla(\psi E)\cdot\nabla\phi\, dy \\
&\le CN\left(\int_{|y'|<N^{-1/2}} |A(y)-A(0)|^2 e^{-2Ny_n} dy\right)^{1/2} \|\nabla\phi\|_{L^2} \\
&\le C\|\nabla\phi\|_{L^2}\left[N^{1/2}\left(\int_{|y'|<N^{-1/2}} |A(y',0)-A(0)|^2\, dy'\right)^{1/2}\right. \qquad (9) \\
&\left. +N\|DF^{-1}\|_\infty^2\left(\int_{\substack{|y'|<N^{-1/2}\\ y_n>0}} |\gamma(F(y',0)-\gamma(F(y',y_n))|^2 e^{-2Ny_n} dy\right)^{1/2}\right].
\end{aligned}
$$

Using (2), (3) and an argument similar to Lemma 1, each of the terms on the right-hand side of the last inequality of (9) is $o(N^{\frac{3-n}{4}})$. Thus we obtain the desired estimate that $\mathrm{III}(\phi) = o(N^{\frac{3-n}{4}})\|\nabla\phi\|_{L^2}$. $\Box$

Given the two Lemmas, the proof of our theorem is easy.

Proof of Theorem. Suppose that $P \in \partial\Omega$, let (x', x_n) and F be as in the definition of a Lipschitz domain and, without loss of generality, assume $F(0) = P$ with $x' = 0$ satisfying (H2) and (3). We let $I = (1 + |\nabla\phi(0)|^2)\int_{\mathbf{R}^{n-1}} \eta(|y'|)^2\, dy'$ where η is as in (4) and let

$$f_N = v_N \circ F^{-1} I^{-1/2} N^{\frac{3-n}{4}}.$$

If u_N solves $\operatorname{div} \gamma\nabla u_N = 0, \quad u_N|_{\partial\Omega} = f_N$, then we have

$$\begin{aligned}
\int_{\partial\Omega} \bar{f}_N \Lambda_\gamma f_N &= \int_\Omega \gamma |\nabla u_N|^2 \\
&= \int_{\tilde{\Omega}} A\nabla \bar{u}_N \circ F \cdot \nabla u_N \circ F \\
&= \int_{\tilde{\Omega}} A\nabla v_N \cdot \nabla \bar{v}_N + o(1) \\
&= \gamma(F(0)) + o(1)
\end{aligned}$$

where v_N is as in (4), the third equality is Lemma 2 and the fourth equality is Lemma 1. $\square$

References

[1] Giovanni Alessandrini. Singular solutions of elliptic equations and the determination of conductivity by boundary measurements. *J. Diff. Equations*, 84:252–272, 1990.

[2] Russell M. Brown and Gunther A. Uhlmann. Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions. *Comm. Partial Differential Equations*, 22(5-6):1009–1027, 1997.

[3] E.B. Davies. *Heat kernels and spectral theory.* Cambridge University Press, 1989.

[4] R. Kohn and M. Vogelius. Determining conductivity by boundary measurements. *Comm. Pure Appl. Math.*, 37:289–298, 1984.

[5] Adrian I. Nachman. Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math. (2)*, 143:71–96, 1996.

[6] R. L. Robertson. Boundary identifiability of residual stress via the Dirichlet to Neumann map. *Inverse Problems*, 13(4):1107–1119, 1997.

[7] Erkki Somersalo, Margaret Cheney, David Isaacson, and Eli Isaacson. Layer stripping: a direct numerical method for impedance imaging. *Inverse problems*, 7:899–926, 1991.

[8] E.M. Stein. *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton NJ, 1970.

[9] J. Sylvester. A convergent layer stripping algorithm for the radial symmetric impedance tomography problem. *Comm. Partial Diff. Eqns.*, 17:1955–1994, 1992.

[10] J. Sylvester and G. Uhlmann. Inverse boundary value problems at the boundary–continuous dependence. *Comm. Pure Appl. Math.*, 61:197–219, 1988.

October 11, 2000