The mixed problem for the Lamé system in a class of Lipschitz domains

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Abstract

We consider the mixed problem for the Lamé system

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u|_D = f_D & \text{on } D \\ \dfrac{\partial u}{\partial \rho} = f_N & \text{on } N \\ (\nabla u)^* \in L^p(\partial\Omega) & \end{cases}$$

in the class of bounded Lipschitz creased domains. Here D and N partition $\partial\Omega$ and $\partial/\partial\rho$ stands for the traction operator. We suppose the Dirichlet data f_D has one derivative in $L^p(D)$ and the traction data f_N is in $L^p(N)$. For p in a small interval containing 2, we find a unique solution to the mixed problem subject to the condition that the non-tangential maximal function of the gradient of the solution is in $L^p(\partial\Omega)$.

1 Preliminaries

The study of the Lamé system of elastostatics, equipped with various boundary conditions (of Dirichlet, Neumann and Mixed type) occupies an important place in the mathematical and engineering literature. A standard reference in this regard is [11].

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Subsequent efforts to increase the range of applicability of the mathematical theory developed in this setting have led to the consideration of more general types of domains, whose boundaries are allowed to contain irregularities. For example, in [5], B. Dahlberg, C. Kenig and G. Verchota were able to establish the well-posedness of the Dirichlet and Neumann problems for the Lamé system in arbitrary Lipschitz domains, with L^2 boundary data.

The aim of this note is to find a solution to the mixed problem for the Lamé system in a certain class of Lipschitz domains. The class of domains we consider has been introduced in the study of the mixed problem for the Laplacian in [1] (see also [12], [14], and [19]). For the case of domains with isolated singularities (such as polygonal and polyhedral domains), the reader is referred to, e.g., [9], [15], [16], and the references therein.

Throughout this paper we will use the convention of summing over repeated indices. Let $u : \Omega \to \mathbb{R}^n$, $u = (u^1, \dots, u^n)$, denote a vector-valued function defined on a bounded open set $\Omega \subset \mathbb{R}^n$. It is convenient to write the Lamé system as $Lu = \mathrm{div}\sigma(u)$, or

$$(Lu)^i = \frac{\partial}{\partial x_\alpha}\sigma^i_\alpha(u), \qquad i = 1, \dots, n. \tag{1.1}$$

Above, $\sigma(u) = (\sigma^i_\alpha(u))_{i,\alpha=1,\dots,n}$ denotes the stress tensor, where

$$\sigma^i_\alpha(u) := a^{ij}_{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}, \tag{1.2}$$

whose norm is defined as $|\sigma(u)|^2 := \sum_{i=1}^n \sum_{\alpha=1}^n |\sigma^i_\alpha(u)|^2$. In the sequel, we shall consider a family of stress tensors for which the coefficients $a^{ij}_{\alpha\beta}$ in (1.2) are given by

$$a^{ij}_{\alpha\beta} = \mu\delta_{ij}\delta_{\alpha\beta} + (\lambda + \mu - r)\delta_{i\alpha}\delta_{j\beta} + r\delta_{i\beta}\delta_{j\alpha}, \tag{1.3}$$

for some $r \in \mathbb{R}$.

Notice that one may add a divergence free tensor to $\sigma(u)$ without changing the system of equations $Lu = 0$, where L is as in (1.1). Next, recall the strain tensor

$$\epsilon(u) := (\epsilon_{ij}(u))_{i,j=1,\dots,n}, \qquad \epsilon_{ij}(u) := \frac{1}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right). \tag{1.4}$$

The constants λ and μ in (1.3) are the Lamé parameters, encoding the elastic characteristics of the body Ω. As is standard, throughout the paper we will require that

$$\mu > 0, \quad \lambda \geq -\frac{2\mu}{n} \quad \text{and} \quad r \in [-\mu, \mu], \tag{1.5}$$

which ensures that the coefficient tensor given in (1.3) is semi-positive definite. That is, $a^{ij}_{\alpha\beta}\zeta^{\alpha}_i\zeta^{\beta}_j \geq 0$ for every $\zeta = (\zeta^{\beta}_j)_{j,\beta=1,\ldots,n} \in \mathbb{R}^{n\times n}$; compare with (1.15) below.

The value $r = \mu$ is of particular interest in applications and gives the standard stress tensor in elasticity (see, e.g., [11]). The value $r = \mu(\lambda+\mu)/(3\mu+\lambda)$ gives rise to the so-called pseudo-stress tensor and is of interest from the point of view of layer potentials [5, 13].

Associated with the coefficients (1.3), consider the first-order boundary operator $\partial/\partial\rho$ defined by

$$\left(\frac{\partial u}{\partial \rho}\right)^{\alpha} := \sigma^i_{\alpha}(u)\nu_i, \quad \alpha = 1,\ldots,n. \tag{1.6}$$

In this definition, ν is the outward unit normal vector to $\partial\Omega$. When $r = \mu$ in (1.3), the operator $\partial/\partial\rho$ is called the traction conormal.

The boundary value problem we consider in this paper is

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u|_D = f_D & \text{on } D \\ \dfrac{\partial u}{\partial \rho} = f_N & \text{on } N \\ (\nabla u)^* \in L^p(\partial\Omega). \end{cases} \tag{1.7}$$

We assume above that $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, is a bounded Lipschitz domain with connected boundary. This means that Ω is a bounded open set in $\mathbb{R}^n$, $\partial\Omega$ is connected, and there exists $M > 0$ such that, for each $x \in \partial\Omega$ one may find a coordinate system (obtained by translating and rotating the standard coordinate system in $\mathbb{R}^n$) say, $(x', x_n) = (x_1, x'', x_n) \in \mathbb{R}\times\mathbb{R}^{n-2}\times\mathbb{R}$, a cylinder $C_r(x) := \{(y', y_n) : |y'-x'| < r, |y_n - x_n| < 2Mr\}$ for some $r > 0$, and a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\|\nabla\phi\|_\infty \leq M$ and so that

$$\begin{aligned} C_r(x) \cap \Omega &= \{(y', y_n) : y_n > \phi(y')\} \cap C_r(x), \\ C_r(x) \cap \partial\Omega &= \{(y', y_n) : y_n = \phi(y')\} \cap C_r(x). \end{aligned} \tag{1.8}$$

Above $\|\cdot\|_\infty$ denotes the essential supremum norm. Suppose also that $\partial\Omega = D \cup N$ where D is relatively open in $\partial\Omega$ and $N = \partial\Omega \setminus \bar{D}$. Additional conditions will be imposed below. Hereafter $L^p(\partial\Omega)$, $1 < p < \infty$, stands for the Lebesgue scale of p-integrable functions with respect to the surface measure dS on $\partial\Omega$. For a function w defined in Ω, and any $y \in \partial\Omega$, we define the non-tangential maximal function of w evaluated at y by setting

$$w^*(y) := \sup\{|w(x)| : x \in \Gamma(y)\}, \tag{1.9}$$

where, for $\kappa > 0$ fixed, $\Gamma(y)$ stands for the non-tangential approach region with vertex at $y \in \partial\Omega$ given by

$$\Gamma(y) := \{x \in \Omega : |x - y| < (1 + \kappa)\mathrm{dist}(x, \partial\Omega)\}. \tag{1.10}$$

The trace $u|_D$ is understood in the non-tangential sense, i.e.,

$$u|_D(y) := \lim_{\substack{x \to y \\ x \in \Gamma(y)}} u(x), \qquad \text{for a.e.} \quad y \in D. \tag{1.11}$$

This will be tacitly assumed throughout the paper when dealing with boundary traces of functions defined in Ω.

In (1.7) we take $f_N \in L^p(N)$ and $f_D \in L^{p,1}(D)$, where the latter space denotes the Sobolev space of p integrable functions on D which have one (tangential) derivative in $L^p(D)$ and $\partial/\partial\rho$ is as in (1.6). The conditions on the coefficients (1.3) and (1.5) guarantee that the operator L is elliptic as we shall see below.

Recall that a second order differential operator

$$(Lu)^i := a^{ij}_{\alpha\beta}\partial_i\partial_j u^\beta, \qquad i \in \{1, \dots, n\}, \tag{1.12}$$

is said to satisfy the Legendre-Hadamard ellipticity condition if there is a constant $\gamma > 0$ such that

$$a^{ij}_{\alpha\beta}\xi_i\xi_j\eta^\alpha\eta^\beta \geq \gamma|\xi|^2|\eta|^2, \qquad \text{for all } \xi = (\xi_i)_{i=1,\dots,n} \in \mathbb{R}^n, \eta = (\eta^\alpha)_{\alpha=1,\dots,n} \in \mathbb{R}^n. \tag{1.13}$$

It is easy to check that the coefficients introduced in (1.3) satisfy

$$a^{ij}_{\alpha\beta}\xi_i\xi_j\eta^\alpha\eta^\beta = \mu|\xi|^2|\eta|^2 + (\lambda + \mu)(\xi \cdot \eta)^2. \tag{1.14}$$

Thus (1.13) holds (for some $\gamma > 0$) whenever $\mu > 0$ and $\lambda + 2\mu > 0$.

Going further, the operator L in (1.12) is said to be strongly elliptic provided that there exists a constant $\gamma > 0$ such that

$$a^{ij}_{\alpha\beta}\zeta^\alpha_i\zeta^\beta_j \geq \gamma|\zeta|^2, \qquad \text{for all} \qquad \zeta = (\zeta^\beta_j)_{j,\beta=1,\dots,n} \in \mathbb{R}^{n\times n}. \tag{1.15}$$

For the coefficients defined in (1.3), we have

$$a^{ij}_{\alpha\beta}\zeta^\alpha_i\zeta^\beta_j \geq (\mu - |r| + n(\lambda + \mu - r)^-)|\zeta|^2, \tag{1.16}$$

where $x^- := \min\{0, x\}$ is the negative part of x. Thus if $\lambda + \mu - r \geq 0$ and $-\mu < r < \mu$, the property (1.15) holds with constant $\gamma = \mu - |r|$. The endpoint $r = \mu$, when the strong ellipticity condition (1.15) fails, corresponds to the interesting traction boundary condition. However, for $-\mu < r \leq \mu$ the following weaker ellipticity property holds. There exists $\gamma > 0$ such that for every $\zeta = (\zeta_i^\alpha)_{i,\alpha=1,\dots,n} \in \mathbb{R}^{n\times n}$,

$$a_{\alpha\beta}^{ij}\zeta_\alpha^i\zeta_\beta^j \geq \gamma \left|\frac{\zeta+\zeta^t}{2}\right|^2, \tag{1.17}$$

where the superscript t indicates transposition. Indeed,

$$\begin{aligned} a_{\alpha\beta}^{ij}\zeta_\alpha^i\zeta_\beta^j &= \mu|\zeta|^2 + (\lambda+\mu-r)(\zeta_i^i)^2 + r\zeta_\alpha^i\zeta_i^\alpha \\ &= 2r\left|\frac{\zeta+\zeta^t}{2}\right|^2 + (\mu-r)|\zeta|^2 + (\lambda+\mu-r)(\zeta_i^i)^2 \qquad (1.18) \\ &\geq (\mu+r)\left|\frac{\zeta+\zeta^t}{2}\right|^2 + (\mu-r)\left|\frac{\zeta-\zeta^t}{2}\right|^2 + n(\lambda+\mu-r)^-\left|\frac{\zeta+\zeta^t}{2}\right|^2. \end{aligned}$$

For the mixed problem treated in this note, an additional feature of our domain is important. Recall the sets N and D on which the Dirichlet and Neumann data are specified. The additional requirement is that there is a constant $m > 0$ such that, if $x \in \bar{D} \cap \bar{N}$, there is a Lipschitz function $\psi : \mathbb{R}^{n-2} \to \mathbb{R}$ and $r > 0$ satisfying

$$\begin{aligned} N \cap C_r(x) &= C_r(x) \cap \partial\Omega \cap \{(x_1, x'', x_n) : \, x_1 \geq \psi(x'')\}, \\ D \cap C_r(x) &= C_r(x) \cap \partial\Omega \cap \{(x_1, x'', x_n) : \, x_1 < \psi(x'')\}, \end{aligned} \tag{1.19}$$

and for which $\frac{\partial\phi}{\partial x_1} > m$, a.e. when $x_1 > \psi(x'')$ and $\frac{\partial\phi}{\partial x_1} < -m$ a.e. when $x_1 < \psi(x'')$. Here ϕ is as in (1.8). As $\partial\Omega$ is compact, we may find $\delta > 0$ and a finite collection of cylinders $\{C_1, \dots, C_N\}$ which cover $\partial\Omega$ and such that in each coordinate cylinder we have a unit vector h_i with $h_i \cdot \nu > \delta$ a.e. on $C_i \cap N$ and $h_i \cdot \nu < -\delta$ a.e. on $C_i \cap D$. Here 'dot' denotes the scalar product in $\mathbb{R}^n$. Using a partition of unity, we may patch the h_i's together to obtain a vector field $h \in C^\infty(\bar{\Omega})$ which satisfies

$$\begin{aligned} h\cdot\nu > \delta \qquad &\text{a.e. on } N, \\ h\cdot\nu < -\delta \qquad &\text{a.e. on } D. \end{aligned} \tag{1.20}$$

We call such domains creased domains. The reason for this terminology is the fact that the condition (1.20) requires that the domain be non-smooth along the interface between D and N. In the work of one of the authors [1] it is shown that one may solve the mixed problem with data in L^2 for Laplace's equation in these domains.

Finally, we define the tangential gradient (on $\partial\Omega$) of a scalar-valued function u by

$$\nabla_t u := \nabla u - (\nu\cdot\nabla u)\nu, \tag{1.21}$$

and agree that ∇_t acts on vector fields component-wise.

2 The Main Result

We are ready to state the main theorem of this paper.

Theorem 2.1 *Let Ω be a bounded Lipschitz creased domain in $\mathbb{R}^n$, $n \geq 3$ and let L be as in (1.1), where the parameters λ, μ and r are as in (1.5). Then there exists $\varepsilon > 0$ such that for $|p-2| < \varepsilon$ the mixed problem (1.7) has a unique solution u.*

Moreover, for p as above, there exists $C = C(\Omega, D, N, \mu, \lambda, r, p) > 0$ such that for any $f_D \in L^{p,1}(D)$ and $f_N \in L^p(N)$ the solution u of (1.7) satisfies

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\Big(\|f_N\|_{L^p(N)} + \|f_D\|_{L^{p,1}(D)}\Big). \tag{2.1}$$

A key step in the proof of Theorem 2.1 is establishing an estimate at the boundary for solutions of (1.7) with $p = 2$ in bounded Lipschitz creased domains in $\mathbb{R}^n$ with $n \geq 3$. This will be achieved with the help of a sequence of lemmas. This portion of our work builds on certain results of B. Dahlberg, C. Kenig and G. Verchota [5] which we now recall.

Let us start by writing out a version of the Rellich identity for systems. To this end consider a general second order, constant coefficient differential operator in $\mathbb{R}^n$,

$$(Lu)^\alpha := \frac{\partial}{\partial x_i} a^{ij}_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j}, \qquad \alpha = 1, \ldots, n, \tag{2.2}$$

whose coefficients $a^{ij}_{\alpha\beta}$ satisfy the symmetry condition

$$a^{ij}_{\alpha\beta} = a^{ji}_{\beta\alpha}. \tag{2.3}$$

The Rellich identity in the next Lemma is due to L. Payne and F. Weinberger [17]. For related work see also [5].

Lemma 2.2 *Let Ω be a bounded Lipschitz domain, and $h = (h_k)_{1\leq k\leq m}$ be a smooth vector field with components in $C^\infty(\bar{\Omega})$. Suppose that L is as in (2.2), with the coefficients $a^{ij}_{\alpha\beta}$ satisfying (2.3), and let u be a solution of $Lu = 0$ in Ω with $(\nabla u)^* \in L^2(\partial\Omega)$. Then the following identity holds*

$$\begin{aligned}\int_{\partial\Omega} &\left(h_k \nu_k a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} - 2\nu_i a^{ij}_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} h_k \frac{\partial u^\alpha}{\partial x_k} \right) dS \\ &= \int_{\Omega} \left(\frac{\partial h_k}{\partial x_k} a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} - 2 \frac{\partial h_k}{\partial x_i} a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_k} \frac{\partial u^\beta}{\partial x_j} \right) dx. \end{aligned} \tag{2.4}$$

Proof. When the domain Ω is smooth and $u \in C^\infty(\bar{\Omega})$ the proof given in [17] is a computation using the divergence theorem. For the more general setting considered here one can use the identity just discussed in a sequence of smooth approximating domains $\Omega_j \subseteq \Omega$ such that Ω_j increases to Ω along with the existence of the trace $u|_{\partial\Omega}$ and the Lebesgue Dominated Convergence Theorem. ∎

Next, we recall the Korn inequality as presented, for example, by P. Ciarlet in [3]. With the notation introduced in (1.4) we have

Theorem 2.3 *Let Ω be a bounded Lipschitz domain with connected boundary and $D \subset \partial\Omega$ a set of positive surface measure. Then there exists $C = C(\partial\Omega, D) > 0$ such that*

$$\int_\Omega \left[|u|^2 + |\nabla u|^2\right] dx \le C \left(\int_\Omega |\epsilon(u)|^2 \, dx + \int_D |u|^2 \, dS \right), \tag{2.5}$$

for all $u \in L^{2,1}(\Omega)$.

We also state a version of Poincaré's inequality on the boundary to the effect that if Ω is as in the statement of Theorem 2.3 and $D \subset \partial\Omega$ has positive surface measure, there exists $C = C(\partial\Omega, D) > 0$ such that

$$\int_{\partial\Omega} |u|^2 \, dS \le C \left(\int_{\partial\Omega} |\nabla_t u|^2 \, dS + \int_D |u|^2 \, dS \right), \tag{2.6}$$

for each $u \in L^{2,1}(\partial\Omega)$.

The main estimate for the mixed problem (1.7) with L^2 data is established next.

Theorem 2.4 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$, be a bounded Lipschitz creased domain with connected boundary and recall the partition of $\partial\Omega$ as $\partial\Omega = D \cup N$. Assume that L is a second order differential operator as in (2.2) whose coefficients satisfy the symmetry condition (2.3), and either: (i) the coefficients $a^{ij}_{\alpha\beta}$ are defined by (1.3) and with parameters μ, λ and r as in (1.5), or: (ii) that the operator L is strongly elliptic (see (1.15)). Then there exists $C = C(L, \partial\Omega, D, N) > 0$ such that the estimate*

$$\int_{\partial\Omega} |\nabla u|^2 \, dS \le C \left(\int_N \left| \frac{\partial u}{\partial \rho} \right|^2 \, dS + \int_D \left[|\nabla_t u|^2 + |u|^2 \right] \, dS \right) \tag{2.7}$$

holds whenever u satisfies $Lu = 0$ in Ω and $(\nabla u)^* \in L^2(\partial\Omega)$.

Remark. Similar results hold in the case of graph Lipschitz domains under the assumption that D, N form a creased partition of $\partial\Omega$. We leave the details to the interested reader.

In preparation for presenting the proof of Theorem 2.4 we record the following interesting result of B. Dahlberg, C. Kenig and G. Verchota (Theorem 1.23 in [5]). We will use a minor variation of their result and include a proof to show that our formulation follows easily from theirs.

Lemma 2.5 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that L is an elliptic second order differential operator as in (2.2) whose coefficients satisfy (2.3) and (1.17). Then there exists $C = C(L, \partial\Omega) > 0$ such that for any solution u of the operator L in Ω such that $(\nabla u)^* \in L^2(\partial\Omega)$, there holds*

$$\int_{\partial\Omega} |\nabla u|^2 \, dS \leq C \left(\int_{\partial\Omega} |\epsilon(u)|^2 \, dS + \int_{\partial\Omega} |u|^2 \, dS \right). \tag{2.8}$$

Proof. According to Theorem 1.23 of [5], there exists $C > 0$ such that

$$\int_{\partial\Omega} |\nabla u|^2 \, dS \leq C \left(\int_{\partial\Omega} |\epsilon(u)|^2 \, dS + \int_{\Omega} \left[|\nabla u|^2 + |u|^2 \right] \, dx \right), \tag{2.9}$$

for all u such that $Lu = 0$ in Ω and $(\nabla u)^* \in L^2(\partial\Omega)$. Using this and (2.5), we obtain

$$\int_{\partial\Omega} |\nabla u|^2 \, dS \leq C \left(\int_{\Omega} |\epsilon(u)|^2 \, dx + \int_{\partial\Omega} |\epsilon(u)|^2 \, dS + \int_{\partial\Omega} |u|^2 \, dS \right). \tag{2.10}$$

Finally, the ellipticity condition (1.17) and integration by parts implies

$$\begin{aligned} \int_{\Omega} |\epsilon(u)|^2 \, dx &\leq C \int_{\Omega} a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} \, dx = C \int_{\partial\Omega} u \cdot \frac{\partial u}{\partial \rho} \, dS \\ &\leq C \int_{\partial\Omega} \left[\theta \left| \frac{\partial u}{\partial \rho} \right|^2 + \theta^{-1} |u|^2 \right] dS, \qquad \theta > 0. \end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11) and choosing $\theta > 0$ small enough gives (2.8) as desired. ∎

With this in hand, we now turn to the proof of the main estimate (2.7).

Proof of Theorem 2.4. We treat the assumption (i) in the statement of the theorem (i.e., the case when the operator L is as in (1.1), with the coefficients from (1.3) having the parameters μ, λ and r as in (1.5)). The proof is simpler when the operator L is strongly elliptic (the assumption (ii) in the statement of the Theorem) and we omit the details in this case.

The starting point is the identity (2.4) of Lemma 2.2 which readily gives

$$\int_{\partial\Omega} h_k \nu_k a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j}\, dS = 2\int_{\partial\Omega} \nu_i a^{ij}_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} h_k \frac{\partial u^\alpha}{\partial x_k}\, dS \tag{2.12}$$
$$+ \int_\Omega \left(\frac{\partial h_k}{\partial x_k} a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} - 2a^{ij}_{\alpha\beta} \frac{\partial h_k}{\partial x_i} \frac{\partial u^\alpha}{\partial x_k} \frac{\partial u^\beta}{\partial x_j} \right) dx.$$

Subtracting the quantity

$$2\int_D h_k \nu_k a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j}\, dx \tag{2.13}$$

from both sides of (2.12) yields

$$\int_N h_k \nu_k a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j}\, dS - \int_D h_k \nu_k a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j}\, dS$$
$$= 2\int_N \nu_i a^{ij}_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} h_k \frac{\partial u^\alpha}{\partial x_k}\, dS + 2\int_D a^{ij}_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} h_k \left(\nu_i \frac{\partial u^\alpha}{\partial x_k} - \nu_k \frac{\partial u^\alpha}{\partial x_i} \right) dS$$
$$+ \int_\Omega \left(\frac{\partial h_k}{\partial x_k} a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} - 2a^{ij}_{\alpha\beta} \frac{\partial h_k}{\partial x_i} \frac{\partial u^\alpha}{\partial x_k} \frac{\partial u^\beta}{\partial x_j} \right) dx. \tag{2.14}$$

Going further, we may use (1.17) and the properties of the vector field h (see (1.20)) to obtain the following lower bound for the left-hand side of (2.14):

$$c\int_{\partial\Omega} |\epsilon(u)|^2\, dS \le \int_N h_k \nu_k a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j}\, dS - \int_D h_k \nu_k a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j}\, dS. \tag{2.15}$$

Consider next the terms on the right-hand side of (2.14). For the first term, we use the Cauchy-Schwarz inequality to write

$$\int_N \nu_i a^{ij}_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} h_k \frac{\partial u^\alpha}{\partial x_k}\, dx \le \int_N \left[C\theta^{-1} \left| \frac{\partial u}{\partial \rho} \right|^2 + \theta |\nabla u|^2 \right] dS. \tag{2.16}$$

For the second term in the right-hand side of (2.14), we observe that $\nu_i \frac{\partial u^\alpha}{\partial x_k} - \nu_k \frac{\partial u^\alpha}{\partial x_i}$ is a tangential derivative of u^α. Thus, we may use the Cauchy-Schwarz inequality to obtain that

$$\int_D a^{ij}_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} h_k \left(\nu_i \frac{\partial u^\alpha}{\partial x_k} - \nu_k \frac{\partial u^\alpha}{\partial x_i} \right) dS \le \int_D \left[C\theta |\nabla u|^2 + \theta^{-1} |\nabla_t u|^2 \right] dS. \tag{2.17}$$

To estimate the integral over Ω in (2.14), we use that h is smooth, and (2.5) from Theorem 2.3. Hence,

$$\int_\Omega \left(\frac{\partial h_k}{\partial x_k} a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} - 2 a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_k} \frac{\partial h_k}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} \right) dx$$
$$\leq C \int_\Omega |\nabla u|^2 \, dx$$
$$\leq C \left(\int_\Omega |\epsilon(u)|^2 \, dx + \int_D |u|^2 \, dS \right). \tag{2.18}$$

Next, using (1.17), we write

$$\begin{aligned} \int_\Omega |\epsilon(u)|^2 \, dx &\leq C \int_\Omega a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} \, dx = C \int_{\partial\Omega} u \cdot \frac{\partial u}{\partial \rho} \, dS \\ &\leq C \left(\int_D \left[\theta^{-1} |u|^2 + \theta \left| \frac{\partial u}{\partial \rho} \right|^2 \right] dS + \int_N \left[\theta |u|^2 + \theta^{-1} \left| \frac{\partial u}{\partial \rho} \right|^2 \right] dS \right). \end{aligned} \tag{2.19}$$

Combining (2.15) with (2.18) and (2.19), we conclude that for each $\theta > 0$ the following holds

$$\begin{aligned} c \int_{\partial\Omega} |\epsilon(u)|^2 \, dS &\leq \int_\Omega \Big(\frac{\partial h_k}{\partial x_k} a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} - 2 a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial h_j}{\partial x_j} \frac{\partial u^\beta}{\partial x_j} \Big) \, dx \\ &\leq C \left(\int_D \left[(1 + \theta^{-1}) |u|^2 + \theta \left| \frac{\partial u}{\partial \rho} \right|^2 \right] dS + \int_N \left[\theta |u|^2 + \theta^{-1} \left| \frac{\partial u}{\partial \rho} \right|^2 \right] dS \right). \end{aligned} \tag{2.20}$$

Unfortunately, the integral in the left-most side of (2.20) does not yet involve the full gradient of u which prevents us from hiding the terms with θ small from the right-most side of (2.20). This is where the boundary Korn inequality (2.8) is needed and this allows us to estimate the full gradient of u at the cost of a term involving the L^2-norm of u. Therefore matters reduce to handling the term $\int_{\partial\Omega} |u|^2 \, dx$. To this end, we write

$$\begin{aligned} \int_{\partial\Omega} |u|^2 \, dx &\leq C \int_\Omega \Big[\nabla u|^2 + |u|^2 \Big] \, dx \\ &\leq C \left(\int_\Omega |\epsilon(u)|^2 \, dx + \int_D |u|^2 \, dS \right) \\ &\leq C \left(\int_\Omega a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} \, dx + \int_D |u|^2 \, dS \right) \end{aligned} \tag{2.21}$$

$$
\begin{aligned}
&= C\left(\int_{\partial\Omega} u\cdot\frac{\partial u}{\partial\rho}\,dS + \int_D |u|^2\,dS\right)\\
&\leq C\left(\int_N \left[\theta^{-1}\left|\frac{\partial u}{\partial\rho}\right|^2 + \theta|u|^2\right]dS + \int_D\left[(1+\theta^{-1})|u|^2 + \theta\left|\frac{\partial u}{\partial\rho}\right|^2\right]dS\right)\\
&\leq C\left(\int_N \theta^{-1}\left|\frac{\partial u}{\partial\rho}\right|^2\,dS + \int_D\left[\theta\left|\frac{\partial u}{\partial\rho}\right|^2 + (1+\theta+\theta^{-1})|u|^2\right]dS\right)\\
&\quad +C\int_{\partial\Omega}\theta|\nabla_t u|^2\,dS.
\end{aligned}
$$

In (2.21), the first inequality uses the boundedness of the trace operator as a map from $L^{2,1}(\Omega)$ to $L^2(\partial\Omega)$, the second inequality is a consequence of the Korn inequality (2.5) in Ω, and the third one follows from (1.17). Going further, the equality in (2.21) follows from integration by parts, the fourth inequality is a simple application of Cauchy-Schwarz (and splitting the domain of integration over the boundary pieces N and D), and the last inequality is a consequence of the Poincaré inequality (2.6). Finally (2.7) follows by choosing $\theta > 0$ small enough. ∎

Consider next the Dirichlet-to-Neumann map Λ associated with the Lamé operator L introduced in (1.1) given by

$$
\Lambda f := \frac{\partial u}{\partial\rho} \tag{2.22}
$$

where $\partial/\partial\rho$ is the conormal introduced in (1.6) associated with the coefficients (1.3) and u is the solution of the Dirichlet problem

$$
\begin{cases}
Lu = 0 & \text{in } \Omega\\
u = f & \text{on } \partial\Omega\\
(\nabla u)^* \in L^p(\partial\Omega).
\end{cases} \tag{2.23}
$$

If the data f is in $L^{p,1}(\partial\Omega)$ for p near 2 and $r\in(-\mu,\mu]$, we know from [5] that there exists a solution to (2.23). Consequently, the map

$$
\Lambda : L^{p,1}_0(N) \to L^p(N) \tag{2.24}
$$

exists and is continuous for $p\in(2-\varepsilon, 2+\varepsilon)$ for some $\varepsilon > 0$. To be more precise, let $L^{p,1}_0(N)$ denote the Sobolev space of p-integrable functions on the interior of N with one derivative in $L^p(N)$ and which vanish on the boundary of N. For $f\in L^{p,1}_0(N)$, let $\tilde f\in L^{p,1}(\partial\Omega)$ be the extension by zero of f to $\partial\Omega$ and consider u the solution of the

Dirichlet problem (2.23) with data $\tilde{f}$. In this notation we set $\Lambda f := \left.\frac{\partial u}{\partial \rho}\right|_N$. We may make a similar definition of Λ for a general strongly elliptic system, see W. Gao [7].

The solvability of the mixed problem can then be formulated using the Dirichlet-to-Neumann map (2.24). If we can solve the Dirichlet problem when the data f is in $L^{p,1}(\partial\Omega)$, then the existence of a solution for the mixed problem is equivalent to the surjectivity of the map (2.24).

Remark. Strictly speaking, B. Dahlberg, C. Kenig and G. Verchota [5] treat (2.23) only for the case $p = 2$, in which scenario the authors establish the invertibility of certain singular integral operators of layer potential type in $L^2(\partial\Omega)$. Known perturbation arguments (see, e.g., [18], [20]) permit one to extend such invertibility results to the $L^p(\partial\Omega)$ scale with p near 2. In turn, this ultimately leads to an extension of the main well-posedness results in [5] to the L^p setting with p near 2.

Lemma 2.6 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and recall the partition of $\partial\Omega = D \cup N$. Suppose that L is an operator as in (2.2) and the coefficients $a^{ij}_{\alpha\beta}$ satisfy either of the conditions in Theorem 2.4. Suppose u is a solution of the mixed problem (1.7) with L^p data with $p \geq 2 - \frac{1}{n}$ if $n \geq 3$ or $p > 1$ if $n = 2$. If the surface measure of D is not zero and $f_N = 0$ and $f_D = 0$, then $u = 0$.*

Proof. Adapting the argument in [2] (see [21] for a correction) one can show that

$$(\nabla u)^* \in L^p(\partial\Omega) \;\Rightarrow\; u^* \in L^q(\partial\Omega) \quad \text{if} \; \begin{cases} 1/q = 1/p - 1/(n-1) & \text{if } n \geq 3, \\ 1/q = 1/p' = 1 - 1/p & \text{if } n = 2. \end{cases} \tag{2.25}$$

Using a family of Lipschitz domains which increase to Ω, we may apply the divergence theorem and obtain

$$\int_\Omega a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j}\, dx = \int_{\partial\Omega} u \frac{\partial u}{\partial \rho}\, dS = 0. \tag{2.26}$$

A key observation in establishing (2.26) is that our hypotheses guarantee that $u^* \in L^{p'}(\partial\Omega)$ with $1/p + 1/p' = 1$. When the strong ellipticity condition (1.15) is satisfied, then (2.26) and the Poincaré inequality imply that $u = 0$. This uses that the surface measure of D is positive. Finally, if instead of (1.15) the weaker ellipticity condition in (1.17) holds, then one must use the Korn inequality (2.5) also in order to conclude that $u = 0$. ∎

Theorem 2.7 *Let Ω be a bounded, creased, Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$, and assume that λ, μ and r are as in (1.5). If Λ is the Dirichlet-to-Neumann map from (2.24) associated with the Lamé system (1.1) and the conormal derivative from (1.6), then there exists $\varepsilon = \varepsilon(\Omega, D, N, \lambda, \mu, r) > 0$ such that $\Lambda : L^{p,1}_0(N) \to L^p(N)$ is an isomorphism for $|p-2| < \varepsilon$.*

Proof. Fix λ, μ and r as in (1.5) and note that Lemma 2.6 gives that the map Λ is injective for p in an open interval containing 2. To show that the map is onto, we use the uniform estimate of Theorem 2.4 and the method of continuity. We will consider two one-parameter families of operators which connect the operator Λ with the Dirichlet-Neumann map for the Laplacian acting on vector-valued functions, Λ_0. As an intermediate step, we will need to consider the operator Λ_1 which is associated with the operator (1.1) and the choice of coefficients given by (1.3) when $r = 0$.

We begin by defining the family of operators L_t with coefficients

$$a^{ij}_{\alpha\beta} = \mu\delta_{ij}\delta_{\alpha\beta} + t\delta_{i\alpha}\delta_{j\beta}, \quad 0 \leq t \leq \lambda + \mu. \tag{2.27}$$

Let $\Lambda(t)$ denote the Dirichlet-to-Neumann map for L_t. At $t = 0$, the operator $\Lambda(0) = \Lambda_0$ is the Dirichlet to Neumann map for the Laplacian, extended in the obvious way to vector-valued functions. According to [1], the map Λ_0 is invertible. For $0 \leq t \leq \lambda + \mu$, Theorem 2.4 implies the estimate

$$\|f\|_{L^{2,1}_0(N)} \leq C\|\Lambda(t)f\|_{L^2(N)}, \quad \text{uniformly for} \quad f \in L^{2,1}_0(N). \tag{2.28}$$

In addition, from the work of R. Coifman, A. McIntosh and Y. Meyer in [4] (see also E. Fabes, M. Jodeit and N. Rivière [6] and S. Hofmann [10]) it follows that the assignment $t \mapsto \Lambda(t)$ is Lipschitz, i.e.

$$\|\Lambda(t)f - \Lambda(s)f\|_{L^2(N)} \leq C|t-s|\|f\|_{L^{2,1}_0(N)}, \tag{2.29}$$

for every $t, s \in [0, \lambda+\mu]$, uniformly in $f \in L^{2,1}_0(N)$. Thus, the method of continuity [8, Theorem 5.2] implies $\Lambda_1 = \Lambda(\lambda+\mu)$ is invertible as an operator from $L^{2,1}_0(N)$ onto $L^2(N)$.

To deform Λ_1 into Λ, we consider the family of operators with coefficients

$$a^{ij}_{\alpha\beta} = \mu\delta_{ij}\delta_{\alpha\beta} + (\lambda+\mu-t)\delta_{i\alpha}\delta_{j\beta} + t\delta_{i\beta}\delta_{j\alpha}, \quad \begin{cases} 0 \leq t \leq r & \text{if } r \in [0,\mu], \\ r \leq t \leq 0 & \text{if } r \in (-\mu, 0). \end{cases} \tag{2.30}$$

If we now let $\Lambda(t)$ denote the Dirichlet to Neumann map associated with this family of coefficients, we have that the family of operators $\Lambda(t)$ associated with (2.30) is continuous and satisfies the estimate (2.28) and the continuity estimate (2.29). As the operator $\Lambda(t)$ is invertible for $t = 0$ by the above, the method of continuity implies the operator is also invertible at $t = r$ and hence the map $\Lambda = \Lambda(r)$ is invertible when $p = 2$.

Since due to the remark above there exists $\varepsilon > 0$ so that the map Λ in (2.24) is continuous whenever $|p-2| < \varepsilon$, we invoke again the perturbation arguments of [18] to conclude that the map Λ will be invertible whenever p is in some open interval which contains 2. ∎

We can finally give the proof of our main theorem.

Proof of Theorem 2.1. As noted earlier, we can solve the Dirichlet problem with data in $L^{p,1}(\partial\Omega)$ for p in a neighborhood of 2. Given this result, the existence of solutions follows from Lemma 2.7 which gives the invertibility of the Dirichlet-to-Neumann map acting on $L_0^{p,1}(N)$. The uniqueness of solutions is given in Lemma 2.6. ∎

Remark. The well-posedness of (1.7) implies, a posteriori, that the solution can be represented in the form of elastic layer potentials. More specifically, let $\mathcal{S}$ be the single layer potential operator (mapping fields on $\partial\Omega$ into fields in Ω), and denote by S its trace to the boundary. Then (see, e.g., [5]) $\partial/(\partial\rho) \circ \mathcal{S} = -\frac{1}{2}I + K^*$ where I is the identity and K^* is the so-called (adjoint) boundary double layer. Then the fact that (1.7) is uniquely solvable becomes equivalent to the invertibility of the assignment

$$T : L^p(\partial\Omega) \to L^{p,1}(D) \oplus L^p(N), \quad Tg := \Big(\Big[Sg \Big]\Big|_D , \ \Big[(-\tfrac{1}{2}I + K^*)g \Big]\Big|_{\partial\Omega} \Big), \tag{2.31}$$

whenever $|p-2| < \varepsilon$. Consequently, the solution of (1.7) can be represented in the form

$$u = \mathcal{S}\Big(T^{-1}(f_D, f_N) \Big) \qquad \text{in } \ \Omega. \tag{2.32}$$

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