GLOBAL UNIQUENESS IN THE IMPEDANCE IMAGING PROBLEM FOR LESS REGULAR CONDUCTIVITIES*

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Abstract. If $L_\gamma = \operatorname{div} \gamma \nabla$ is an elliptic operator with scalar coefficient γ, we show that we can recover the coefficient γ from the Dirichlet to Neumann map under the assumption that γ has only $3/2 + \epsilon$ derivatives. Previously, the best result required γ to have two derivatives.

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Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$, be a bounded open set and let $L_\gamma = \operatorname{div} \gamma \nabla$ be an elliptic operator on Ω with scalar coefficient γ. We let Λ_γ denote the Dirichlet to Neumann map $\Lambda_\gamma f = \gamma \partial u / \partial \nu$ where u is the solution to the Dirichlet problem $L_\gamma u = 0$ in Ω, $u = f$ on $\partial\Omega$. In 1987, Sylvester and Uhlmann [10] showed that if we restrict attention to γ which are sufficiently smooth, then the map $\gamma \to \Lambda_\gamma$ is injective. Nachman, Sylvester and Uhlmann [8], showed that injectivity continues to hold if γ has two bounded derivatives. Extensions to slightly less smooth conductivities or the related Schrödinger equation are given in Chanillo [3] and Ramm [9]. Isakov [5] has established injectivity for conductivities with jump discontinuities.

Since the only smoothness assumption needed to define Λ_γ is that γ be measurable, it is reasonable to ask if the map $\gamma \to \Lambda_\gamma$ is injective under less restrictive hypotheses on γ. In this paper, we show that γ need have only $\dfrac{3}{2} + \epsilon$ derivatives. There is no reason to believe that the result in this paper is optimal. I conjecture that the right smoothness assumption is that γ have one derivative. However, the methods presented here do not give this. To state our main result, we recall the standard space of Hölder continuous functions $C^\alpha(\bar\Omega) = \{f : \ f : \Omega \to \mathbf{R}$ and $|f(x) - f(y)| \leq M|x-y|^\alpha$ for some $M > 0\}$.

THEOREM 0.1. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$ be a bounded, Lipschitz domain. Then the map $\gamma \to \Lambda_\gamma$ is injective on the set $\{\gamma : \ \gamma > 0$ in $\bar\Omega$, $\ \nabla\gamma \in \cup_{\epsilon > 0} C^{1/2+\epsilon}(\bar\Omega)\}$.*

The outline of our argument is the same as in [10]. We construct special solutions of $L_\gamma u = 0$ by studying a Schrödinger operator $\Delta - q$. The innovation here is that we consider potentials q which lie in a Besov space of negative order.

We begin by recalling the Besov spaces and some of their simple properties. We will use the monograph of Bergh and Löfstrom [2] as our reference for these spaces. For $s \in \mathbf{R}$ and $1 \leq p, q \leq \infty$, we let $B^s_{p,q}$ denote the Besov space of distributions. Roughly speaking, a distribution in $B^s_{p,q}$ has s derivatives in L^p. We recall that if $0 < s < 1$, $1 \leq p, q < \infty$, then $f \in B^s_{p,q}$ if and only if

$$(1) \qquad \|f\|_{L^p} + \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} |f(x+h) - f(x)|^p \, dx \right)^{q/p} |h|^{-n-sq} \, dh \right)^{1/q}$$

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is finite. Furthermore, the expression in (1) gives a norm on $B^s_{p,q}$. When $p = q = \infty$, the limiting version of (1) is

$$\|f\|_{L^\infty} + \sup_{x\in\mathbf{R}^n,\ |h|\neq 0} |h|^{-s}|f(x+h) - f(x)|.$$

This provides a norm for $B^s_{\infty,\infty}$ and thus, for $0 < s < 1$, $B^s_{\infty,\infty} = C^s(\mathbf{R}^n)$.

We also consider a scale of weighted Besov spaces $B^{s,\delta}_{p,q}$ defined for $\delta \in \mathbf{R}$ by

$$B^{s,\delta}_{p,q} = \{f:\ (1+|x|^2)^{\delta/2} f \in B^s_{p,q}\}$$

with the norm

$$\|f\|_{B^{s,\delta}_{p,q}} = \|(1+|x|^2)^{\delta/2} f\|_{B^s_{p,q}}.$$

We will use $B^{s,c}_{p,q}$ to denote the distributions in $B^s_{p,q}$ which are compactly supported and

$$B^{s,\mathrm{loc}}_{p,q} = \{f:\ \psi f \in B^s_{p,q} \text{ for each } \psi \in C^\infty_0(\mathbf{R}^n)\}.$$

We recall that $B^0_{2,2}$ is the usual Lebesgue space L^2 on $\mathbf{R}^n$. If follows that

$$B^{0,\delta}_{2,2} = \{f:\ (1+|x|^2)^{\delta/2} f \in L^2\}$$

is the weighted Lebesgue space L^2_δ used by Sylvester and Uhlmann. We also have that $B^1_{2,2}$ is the Sobolev space of functions having one derivative in L^2 and that

$$B^{1,\delta}_{2,2} = \{f:\ f,\ \ \nabla f \in L^2_\delta\}. \tag{2}$$

Next, we note that since $B^s_{2,2}$ and $B^{s,\delta}_{2,2}$ are isomorphic, we may identify the complex interpolation spaces

$$[B^{s_0,\delta}_{2,2} B^{s_1,\delta}_{2,2}]_\theta = B^{s_\theta,\delta}_{2,2}, \quad 0 < \theta < 1$$

where $s_0,\ \ s_1 \in \mathbf{R},\ \ s_\theta = (1-\theta)s_0 + \theta s_1$ (see [2, Theorem 6.4.5]).

The reason for introducing the Besov spaces to be able to define products of (certain) distributions as bilinear maps between Besov spaces. This depends on the following elementary result regarding multiplication in Besov spaces.

Proposition 0.2. *a) If $\|\phi\|_\infty + \|\nabla\phi\|_\infty \le M$, then for $0 < s < 1$, $\delta \in \mathbf{R},\ \ 1 \le p,\ q \le \infty$*

$$\|\psi u\|_{B^{s,\delta}_{p,q}} \le C\|u\|_{B^{s,\delta}_{p,q}} M$$

where $C = C(n,p,q)$.

b) For $0 < s < 1$,

$$\|uv\|_{B^s_{1,2}} \le C\|u\|_{B^s_{2,2}}\|v\|_{B^s_{2,2}}$$

where $C = C(s,n)$.

We do not prove this Proposition, but note that each result follows easily from the norm for $B^s_{p,q}$ given in (1). I thank Mike Frazier for telling me of part b) of the above Proposition.

Next, we give estimates for the operator G_ζ which is the solution operator to the equation

$$\Delta u + 2\zeta \cdot \nabla u = f$$

where $\zeta \in \mathbf{C}^n$.

We observe that G_ζ defined by

$$G_\zeta f = \left(\frac{\hat{f}}{-|\xi|^2 + 2i\zeta \cdot \xi} \right)^\vee \tag{3}$$

maps from $\mathcal{S}$ to $\mathcal{S}'$. Here we are using the Fourier transform defined by $\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi}\, dx$. In [10], it is shown that if $\zeta \cdot \zeta = 0$, then $G_\zeta : L^2_{\delta+1} \to L^2_\delta, \;\; -1 < \delta < 0$, with the bound

$$\|G_\zeta f\|_{L^2_\delta} \le \frac{C}{|\zeta|} \|f\|_{L^2_{\delta+1}}, \;\; |\zeta| > 1 \text{ and } -1 < \delta < 0. \tag{4}$$

We give a simple extension of this result to obtain mapping properties of G_ζ on $B^{s,\delta}_{2,2}$. Shortly before this paper was written, A. Nachman established related estimates for the operator G_ζ in two dimensions [7, Lemma 1.3].

THEOREM 0.3. *Let $\zeta \in \mathbf{C}^n$ satisfy $\zeta \cdot \zeta = 0$ and $|\zeta| > 1$. Then for $-1 < \delta < 0$ and $0 \le s \le 1/2$, the map G_ζ defined by (3) satisfies*

$$\|G_\zeta f\|_{B^{s,\delta}_{2,2}} \le \frac{C}{|\zeta|^{1-2s}} \|f\|_{B^{-s,\delta+1}_{2,2}}.$$

where $C = C(n, s, \delta)$.

Proof. We choose a function ϕ satisfying $\phi = 1$ on $\{\xi : \; |\xi| \le 4|\zeta|\}$, $\;\operatorname{supp}\phi \subset \{\xi : \; |\xi| < 8|\zeta|\}$ and $|\nabla\phi| \le C/|\zeta|$. For $u \in L^2_\delta$, we define

$$Tu = \nabla[(\phi\hat{u})^\vee].$$

We claim that

$$\|Tu\|_{L^2_\delta} \le C|\zeta|\, \|u\|_{L^2_\delta}, \;\; -1 \le \delta \le 1 \tag{5}$$

When $\delta = 0$, this is elementary since T is a multiplier operator whose symbol is bounded by $C|\zeta|$. To obtain (5) when $\delta = 1$, note that

$$\|\hat{u}\|_{L^2} + \|\nabla\hat{u}\|_{L^2}$$

gives an equivalent norm on the weighted Lebesgue space L^2_1. Now

$$\nabla\widehat{Tu} = i\xi\phi\nabla\hat{u} + \hat{u}\nabla(i\xi\phi)$$

and hence

$$\|\nabla\widehat{Tu}\|_{L^2} \le C(|\zeta| + 1)\, \|u\|_{L^2_1}.$$

If we recall that $|\zeta| \ge 1$, then (5) follows for $\delta = 1$. The estimate (5) follows by duality when $\delta = -1$ and for the remaining values of δ, by interpolation. $\square$

Next, define on operator S by

$$\begin{aligned}(Sf)^\wedge(\xi) &= \frac{i\xi(1-\phi)\hat{f}}{(-|\xi|^2+2i\xi\cdot\zeta)} \\ &= \frac{1}{|\xi|}\psi\hat{f}\end{aligned}$$

where $\psi(\xi) = i\xi|\xi|(1-\phi)/(-|\xi|^2+2i\xi\cdot\zeta)$. The argument used to treat T shows that

$$f \to (\psi\hat{f})^\vee$$

is bounded on L^2_δ, $-1 \le \delta \le 1$, and the norm of this operator is bounded for $|\zeta| \ge 1$.

The fractional integral $f \to (|\xi|^{-1}\hat{f})^\vee$ maps $L^2_{\delta+1}$ to L^2_δ, $-1 < \delta < 0$, by the argument in Lemma 3.1, [10]. This gives

$$\|Sf\|_{L^2_\delta} \le C(n,\delta)\|f\|_{L^2_{\delta+1}}, \quad -1 < \delta < 0.$$

Summarizing, we have $\nabla G_\zeta f = T(G_\zeta f) + Sf$ and hence

$$\begin{aligned}\|\nabla G_\zeta f\|_{L^2_\delta} &\le C|\zeta|\,\|G_\zeta f\|_{L^2_\delta} + C\|f\|_{L^2_{\delta+1}} \\ &\le C\|f\|_{L^2_{\delta+1}}\end{aligned}$$

where the second inequality is (4).

Combining this with (5) and the characterization of $B^{1,\delta}_{2,2}$ in (2) gives

$$\|G_\zeta f\|_{B^{1,\delta}_{2,2}} \le C\|f\|_{B^{0,\delta+1}_{2,2}}, \quad -1 < \delta < 0.$$

By duality, we have

$$\|G_\zeta f\|_{B^{0,\delta}_{2,2}} \le C\|f\|_{B^{-1,\delta+1}_{2,2}}, \quad -1 < \delta < 0.$$

Interpolating between these estimates and (4) gives

$$\|G_\zeta f\|_{B^{s,\delta}_{2,2}} \le \frac{C}{|\zeta|^{1-s}}\|f\|_{B^{0,\delta+1}_{2,2}} \tag{6}$$

and

$$\|G_\zeta f\|_{B^{0,\delta}_{2,2}} \le \frac{C}{|\zeta|^{1-s}}\|f\|_{B^{-s,\delta+1}_{2,2}} \tag{7}$$

where each inequality holds for $0 \le s \le 1$ and $-1 < \delta < 0$. Finally interpolating between (6) and (7) gives the estimate of the Theorem.

If g is a function on $\mathbf{R}^n$ satisfying

$$\lambda^{-1} < g < \lambda \tag{8}$$

for some $\lambda > 0$ and ∇g is bounded and compactly supported, then for $u \in C^\infty(\mathbf{R}^n)$, we may define a distribution $m_q(u)$ by

$$m_q(u)(v) = -\int_{\mathbf{R}^n} \nabla g \cdot \nabla\left(\frac{1}{g}uv\right)\,dx. \tag{9}$$

Formally, $q = g^{-1}\Delta g$ will be the potential in our Schrödinger operator and $m_q(u)$ is the product qu. Our main result on m_q is:

THEOREM 0.4. *Suppose that* g *is defined on* $\mathbf{R}^n$*, satisfies (8) and for some* s, $0 < s < 1$, *and* $M > 0$*, satisfies*

$$\|\nabla g\|_{B^{1-s}_{\infty,2}} \le M \tag{10}$$

$$\text{supp } \nabla g \subset \{x : \ |x| < M\}. \tag{11}$$

Then there exists $C = C(M, \lambda, s)$ *so that the map* m_q *satisfies*

$$\|m_q(u)\|_{B^{-s,\delta+1}_{2,2}} \le C\|u\|_{B^{s,\delta}_{2,2}}.$$

Before presenting the proof of this theorem, we note that if $\psi \in C^\infty_0(\mathbf{R}^n)$, then

$$\|\psi u\|_{B^s_{p,q}} \le C(\psi, \delta, s, p, q)\|u\|_{B^{s,\delta}_{p,q}} \tag{12}$$

and if $u \in B^{s,c}_{p,q}$, with supp $u \subset \{x : \ |x| < R\}$, then

$$\|u\|_{B^{s,\delta}_{2,2}} \le C(R, \delta)\|u\|_{B^s_{2,2}}. \tag{13}$$

In each case, the stated inequality follows by observing that if $\psi \in C^\infty_0(\mathbf{R}^n)$ and $r \in \mathbf{R}$, then $u \to (1 + |x|^2)^r\psi u$ is bounded on each Besov space.

Proof of Theorem 4. We prove the estimate of the theorem for u smooth and then we may extend m_q to $B^{s,\delta}_{2,2}$ by density. Let $\psi = 1$ on supp ∇g with $\psi \in C^\infty_0(\mathbf{R}^n)$. Then we write

$$\begin{aligned} |m_q(u)(\phi)| &= \left| \int \psi\nabla g \cdot \nabla \left(\psi^2 \frac{u\phi}{g}\right) \, dx \right| \\ &\le \|\psi\nabla g\|_{B^{1-s}_{\infty,2}} \|\nabla(\psi^2 g^{-1}u\phi)\|_{B^{s-1}_{1,2}} \end{aligned} \tag{14}$$

We use that $\partial/\partial x_i : B^s_{2,2} \to B^{s-1}_{2,2}$, Proposition 2 (b) and then (13) to obtain

$$\begin{aligned} \|\nabla(\psi^2 g^{-1}u\phi)\|_{B^{s-1}_{1,2}} &\le C\|\psi^2 u\phi\|_{B^s_{1,2}} \\ &\le C\|\psi u\|_{B^s_{2,2}}\|\psi\phi\|_{B^s_{2,2}} \\ &\le C\|u\|_{B^{s,\delta}_{2,2}}\|\phi\|_{B^{s,-\delta-1}_{2,2}} \end{aligned} \tag{15}$$

Using (12) and (15) in (14) gives that

$$|m_q(u)(\phi)| \le C\|\nabla g\|_{B^{1-s}_{\infty,2}}\|u\|_{B^{s,\delta}_{2,2}}\|\phi\|_{B^{s,-\delta-1}_{2,2}}$$

or that $m_q(u)$ is in the dual of $B^{s,-\delta-1}_{2,2}$, $\left(B^{s,-\delta-1}_{2,2}\right)' = B^{-s,\delta+1}_{2,2}$ (see [2, Corollary 6.2.8] for the duals of unweighted Besov spaces). □

Remark: An examination of the above proof shows that in fact we have $m_q : B^{s,loc}_{2,2} \to B^{s,c}_{2,2}$. We will use this in Corollary 6 to define $m_q(1)$.

Our next theorem considers solutions to the equation

$$\Delta\psi + 2\zeta\cdot\nabla\psi - m_q(\psi) = f.$$

THEOREM 0.5. *Let g satisfy (8), (10) and (11) and let $\zeta \in \mathbf{C}^n$ satisfy $\zeta\cdot\zeta = 0$. If $0 < s < 1/2$, $\ -1 < \delta < 0$ and $f \in B_{2,2}^{-s,\delta+1}$, then there exists $C_0 = C_0(\lambda, M, s, \delta, n)$ so that for $|\zeta| > C_0$, there exists a unique solution to*

$$\Delta\psi + 2\zeta\cdot\nabla\psi - m_q(\psi) = f, \quad \psi \in B_{2,2}^{s,\delta} \tag{16}$$

and this solution satisfies

$$\|\psi\|_{B_{2,2}^{s,\delta}} \le \frac{C}{|\zeta|^{1-2s}}\|f\|_{B_{2,2}^{-s,\delta+1}}$$

where $C = C(n, s, \delta, M, \lambda)$.

Proof. Consider the map $\psi \to G_\zeta(m_q(\psi))$. By Theorems 3 and 4, we have

$$\|G_\zeta(m_q(\psi))\|_{B_{2,2}^{s,\delta}} \le \frac{C}{|\zeta|^{1-s}}\|\psi\|_{B_{2,2}^{s,\delta}}$$

Hence, if $|\zeta|$ is sufficiently large, then this map is a contraction on $B_{2,2}^{s,\delta}$.

From the uniqueness of solutions to $\Delta\psi + 2\zeta\cdot\nabla\psi = 0$, $\ \psi \in L_\delta^2$ (see [10, Cor. 3.4], [4, Theorem 7.1.27]), ψ satisfies (16) if and only if

$$\psi = G_\zeta(f) + G_\zeta(m_q(\psi)). \tag{17}$$

and by Theorem 3, $G_\zeta(f) \in B_{2,2}^{s,\delta}$. Thus the contraction mapping principle implies solutions to (17) exist and are unique in $B_{2,2}^{s,\delta}$. □

Now we are ready to return to the study of $L_\gamma = \operatorname{div}\gamma\nabla$. It will be convenient to assume that γ is defined in all of $\mathbf{R}^n$ and satisfies for some $1/2 > \epsilon > 0$ and $R > 0$, $\ \lambda > 1$,

$$\lambda^{-1} < \gamma < \lambda \tag{18}$$

$$\nabla\gamma \in B_{\infty,\infty}^{1/2+2\epsilon,c} \subset B_{\infty,2}^{1/2+\epsilon,c} \quad \text{for some } \epsilon > 0 \tag{19}$$

$$\gamma(x) = 1, \quad \text{if } \ |x| > R \tag{20}$$

The embedding in (19) follows easily from the definition of the $B_{p,q}^s$-norm [2, Definition 6.2.2]. Thus if $g = \sqrt{\gamma}$, $\ g$ satisfies the hypotheses of Theorem 5 with $s = \frac{1}{2} - \epsilon$.

COROLLARY 0.6. *Suppose that γ satisfies (18)-(20) and $\zeta \in \mathbf{C}^n$ satisfies $\zeta\cdot\zeta = 0$ and $|\zeta| > C_0$. Then there exists a solution to $L_\gamma u = 0$ of the form*

$$u(x) = \gamma(x)^{-1/2}(1 + \psi(x))e^{x\cdot\zeta}, \quad \psi \in B_{2,2}^{1/2-\epsilon,\delta}.$$

Furthermore, $D^2 u \in L^2_{\mathrm{loc}}(\mathbf{R}^n)$.

Proof. Given γ, we construct m_q as in (9), with $g = \sqrt{\gamma}$. We let ψ be the solution of

$$\Delta\psi + 2\zeta\cdot\nabla\psi - m_q(\psi) = m_q(1).$$

from Theorem 5. Since $v = e^{x\cdot\zeta}(1+\psi)$ solves $\Delta v - m_q(v) = 0$ in $\mathcal{S}'$ and $m_q(v) \in B^{\epsilon-1/2,c}_{2,2}$, regularity theory for Δ implies $v \in B^{\epsilon+3/2,\mathrm{loc}}_{2,2}$ and in particular, $\nabla v \in L^2_{\mathrm{loc}}$. Then a calculation shows that $u = \gamma^{-1/2}v$ solves $L_\gamma u = 0$, $\nabla u \in L^2_{\mathrm{loc}}$. Finally, since γ is C^1, regularity theory for L_γ implies $\nabla^2 u \in L^2_{\mathrm{loc}}$. □

THEOREM 0.7. *Suppose that $\partial\Omega$ is Lipschitz, and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. If $\nabla\gamma_i \in C^{1/2+2\epsilon}(\bar\Omega)$ for some $\epsilon > 0$, then there exist extensions of γ_i to $\mathbf{R}^n$ so that, with $g_i = \sqrt{\gamma}_i$,*

$$\int_{\mathbf{R}^n} \nabla g_1 \cdot \nabla(g_1^{-1}\phi) = \int_{\mathbf{R}^n} \nabla g_2 \cdot \nabla(g_2^{-1}\phi), \quad \phi \in C_0^\infty(\mathbf{R}^n).$$

Proof. We begin by observing that since $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ and $\partial\Omega$ is Lipschitz, we have $\gamma_1 = \gamma_2$ and $\nabla\gamma_1 = \nabla\gamma_2$ on $\partial\Omega$. This result was proven for smooth conductivities in [6] and for C^1 conductivities in Lipschitz domains by [1]. Thus we may extend γ_1 and γ_2 to $\mathbf{R}^n$ so that $\gamma_1 = \gamma_2$ in $\mathbf{R}^n \setminus \bar\Omega$ and satisfies (18)-(20). □

We let u_1 and u_2, be solutions of $L_{\gamma_i} u_i = 0$, $\nabla u_i \in L^2(\Omega)$, $i = 1, 2$. We let $v_i = \gamma_i^{1/2} u_i$ and obtain

$$\begin{aligned} \int_{\partial\Omega} u_2 \Lambda_{\gamma_1} u_1 &= \int_\Omega \gamma_1 \nabla(\gamma_1^{-1/2} v_1) \cdot \nabla(\gamma_1^{-1/2} v_2)\, dx \\ &= \int_\Omega -\nabla\gamma_1^{1/2} \cdot \nabla(\gamma_1^{-1/2} v_1 v_2) + \nabla v_1 \cdot \nabla v_2\, dx \end{aligned}$$

where have used that $\gamma_1 = \gamma_2$ on $\partial\Omega$ and the second equality depends on the product rule.

Reversing the roles of u_1 and u_2 gives

$$\int_{\partial\Omega} u_1 \Lambda_{\gamma_2} u_2 = \int_\Omega -\nabla\gamma_2^{1/2} \cdot \nabla(\gamma_2^{-1/2} v_1 v_2) + \nabla v_1 \cdot \nabla v_2\, dx.$$

If we subtract there expressions and use that Λ_{γ_2} is a symmetric operator, we have

$$\begin{aligned} \int_{\partial\Omega} u_1(\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) u_2 &= \int_\Omega -\nabla\gamma_1^{1/2} \cdot \nabla(\gamma_1^{-1/2} v_1 v_2) \\ &\quad + \nabla\gamma_2^{1/2} \cdot \nabla(\gamma_1^{-1/2} v_1 v_2)\, dx. \end{aligned} \tag{21}$$

If we assume that u_1 and u_2 and defined in all of $\mathbf{R}^n$, then (21), our assumption that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ and that $\gamma_1 = \gamma_2$ in $\mathbf{R}^n \setminus \bar\Omega$ give

$$0 = \int_{\mathbf{R}^n} -\nabla\gamma_1^{1/2} \cdot \nabla(\gamma_1^{-1/2} v_1 v_2) + \nabla\gamma_2^{1/2} \cdot \nabla(\gamma_2^{-1/2} v_1 v_2)\, dx.$$

To choose u_1 and u_2, we fix $k \in \mathbf{R}^n$ and then note that the argument in [10, p. 157] and the estimate of Theorem 5 allow us to construct sequences $u_1^{(n)}, u_2^{(n)}$ so that $L_{\gamma_i} u_i^{(n)} = 0$ and $v_1^{(n)} \cdot v_2^{(n)} \to e^{ix\cdot k}$ in $B^{1/2-\epsilon,\mathrm{loc}}_{1,2}$ as $n \to \infty$. Hence we conclude that

$$\int \nabla\gamma_1^{1/2} \cdot \nabla(\gamma_1^{-1/2} e^{ix\cdot k}) = \int \nabla\gamma_2^{1/2} \cdot \nabla(\gamma_2^{-1/2} e^{ix\cdot k}), \quad k \in \mathbf{R}^n.$$

This implies the conclusion of the Theorem.

PROPOSITION 0.8. *If the conclusion of Theorem 7 holds, then $g_1 = g_2$.*

Proof. We have

$$\int \nabla g_1 \cdot \nabla \left(\frac{1}{g_1} \phi \right) = \int \nabla g_2 \cdot \nabla \left(\frac{1}{g_2} \phi \right)$$

for all $\phi \in C_0^1(\mathbf{R}^n)$. Replace ϕ by $g_1 g_2 \psi$ and observe that this gives

$$\int g_1 g_2 \nabla (\log\, g_1 - \log\, g_2) \cdot \nabla \psi = 0.$$

In particular if $\psi = \log \dfrac{g_1}{g_2}$ then $\log \dfrac{g_1}{g_2} = 0$ in $\mathbf{R}^n$. $\square$

This proposition amounts to observing that the equality $g_1^{-1} \Delta g_1 = g_2^{-1} \Delta g_2$ implies that div $g_1 g_2 \nabla \log(g_1/g_2) = 0$. I thank J. Tolle for showing me this argument, which is due to G. Alessandrini.

Proof of Theorem 1. Suppose that γ_1 and γ_2 are as in the Theorem and that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then we conclude that $\sqrt{\gamma}_1 = \sqrt{\gamma}_2$ from Theorem 7 and Proposition 8. $\square$

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