THE MIXED PROBLEM FOR LAPLACE'S EQUATION IN A CLASS OF LIPSCHITZ DOMAINS

Russell Brown [1]
Department of Mathematics
University of Kentucky
Lexington, KY 40506-0027
U.S.A.

INTRODUCTION

In this paper, we consider the mixed problem for Laplace's equation in a domain in Ω. We assume that N and D give a decomposition of $\partial\Omega$. By this we mean that $N \cup D = \partial\Omega$ and $N \cap D = \emptyset$. Given functions f on D and g on N, we wish to find a function u which satisfies

$$\begin{cases} \Delta u - k^2 u = 0, & \text{in } \Omega, \\ u = f, & \text{on } D, \\ \partial_\nu u = g, & \text{on } N. \end{cases} \tag{MP}$$

We are using $\partial_\nu u = \nabla u \cdot \nu$ to denote the outer normal derivative on $\partial\Omega$. Note also that for technical reasons explained below, we will consider the family of equations $\Delta u - k^2 u = 0, \quad k \in \mathbf{R}$.

It is well-known that under mild restrictions on f and g, a solution to (MP) can be found with $\nabla u \in L^2(\Omega)$. Here, we are concerned with regularity. Our

[1] Partially supported by the NSF and the Commonwealth of Kentucky through the Kentucky EpSCOR program.

main result considers a class of Lipschitz domains and shows that the solution has $\nabla u \in L^2(\partial\Omega)$ when the data g is in $L^2(N)$ and f is in the Sobolev space $L^{2,1}(D)$. We also obtain nontangential maximal function estimates for the solution. This result requires that D and N meet in an angle which is strictly less than π. See (1.1)-(1.4) for the precise hypotheses. Our main estimate for the mixed problem, Lemma 1.7, uses the Rellich identity to bound the norm of ∇u in $L^2(\partial\Omega)$ by the norm of the Dirichlet data in $L^{2,1}(D)$ and the norm of the Neumann data in $L^2(N)$.

Thus our results are similar to those obtained by Jerison and Kenig for the Dirichlet and Neumann problems [10]. The innovation here is that we apply the Rellich identity (see (1.10) below) with a smooth vector field α so that $|\alpha \cdot \nu| \geq \delta > 0$ a.e. on $\partial\Omega$, but $\alpha \cdot \nu$ changes sign as we move from D to N. This can only happen when the normal is discontinuous. Thus the arguments presented below are possible *only* in nonsmooth domains. Furthermore, examples presented at the end of section 2 show that this is an essential restriction. These examples give a family of domains where either our Theorem 2.1 provides the existence and regularity of a solution or it can be shown that the estimates of Theorem 2.1 fail. This negative result leads to one interesting technical problem. When studying boundary value problems on Lipschitz domains, a standard approach (see [9, 16]) has been to approximate a nonsmooth domain by a family of smooth domains where the boundary value problem is well understood. Since the result of Theorem 2.1 is known to fail in smooth domains, this approximation step is unavailable to us. We follow a different approach. To avoid the approximation step, we consider the family of equations $\Delta - k^2$, $k \in \mathbf{R}$. First, we prove estimates for the mixed problem in a domain which lies above the graph of one Lipschitz function. Here the geometry is sufficiently simple that we can use the continuous dependence of certain operators on the domain to reduce to a particularly simple case which can be solved by symmetry. Next, we use a localization argument to transfer the mixed problem on a bounded domain to a family of mixed problems on a finite collection of graph domains. This reduction is most easily executed for large values of k. To see why this might be the case, recall that the fundamental solution for $\Delta - k^2$ decays like $e^{-k|x|}$ as $|x| \to \infty$. Thus the off-diagonal part of the solution operator should be small when k is large and the problem is more easily localized for large k. We remove the restriction that k is large by using the Fredholm theory.

The ideas used here are borrowed from several sources. The use of the Rellich identity in boundary value problems has a long history, see [3, 9, 12]. However, the application to mixed problems presented here seems to be new. The estimates presented for $\Delta - k^2$, $k \neq 0$, are an adaptation of arguments in [1, 2]. The observation in Lemma 2.2 is an adaptation of ideas in [13, 14].

Finally, we note that much effort has been denoted to the study of the mixed problem in polygonal or polyhedral domains. See Grisvard [7, Chapter 4] and Kondrachev [11] for example. As might be expected, one can obtain estimates in a larger class of spaces for these domains. However, the results of this paper apply to a larger class of domains.

§1 THE MIXED PROBLEM IN GRAPH DOMAINS.

Let $\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ be a Lipschitz function. A graph domain is the set $\Omega = \{(X', X_n) : X_n > \phi(X')\}$. In order to study the mixed problem, we need a number of hypotheses on the boundary between D and N. If $\psi : \mathbf{R}^{n-2} \to \mathbf{R}$ is a Lipschitz function we define N and D by

$$\begin{aligned} N &= \{X : \quad X_1 \geq \psi(X'')\} \cap \partial\Omega \\ D &= \{X : \quad X_1 < \psi(X'')\} \cap \partial\Omega \end{aligned} \tag{1.1}$$

where we are using $(X', X_n) = (X_1, X'', X_n)$ with $X' \in \mathbf{R}^{n-1}, X'' \in \mathbf{R}^{n-2}$ and $X \in \mathbf{R}^n$. When $n = 2$, we make the convention that $\psi(X'')$ is some real number. We assume that there exist $\delta_N \geq 0$ and $\delta_D \geq 0$ with $\delta_N + \delta_D > 0$ and such that ϕ satisfies

$$\begin{aligned} &\phi_{X_1} \geq \delta_N \quad \text{a.e. on } \{X_1 > \psi(X'')\} \\ &\phi_{X_1} \leq -\delta_D \quad \text{a.e. on } \{X_1 < \psi(X'')\}. \end{aligned} \tag{1.2}$$

Finally, we will assume that ϕ and ψ are Lipschitz

$$\|\nabla \phi\|_{L^\infty(\mathbf{R}^{n-1})} \leq M \tag{1.3}$$

$$\|\nabla \psi\|_{L^\infty(\mathbf{R}^{n-2})} \leq M. \tag{1.4}$$

Since we only need the case of k real and nonzero in these unbounded domains, we will confine our treatment to this case. However, with minor modifications these results are true for all $k^2 \in \mathbf{C} \setminus \{x + iy : \quad y = 0, \quad x < 0\}$. Our estimates for solutions of (MP) will be stated using the nontangential maximal function. For a function w defined on Ω, the nontangential maximal function of w, $\quad w^*$, is defined for $P \in \partial\Omega$ by

$$w^*(P) = \sup_{X \in \Gamma_\alpha(P)} |w(X)|$$

where for $\alpha > 0$, the nontangential approach region $\Gamma_\alpha(P)$ is given by

$$\Gamma_\alpha(P) = \{X \in \Omega : \ |X - P| < (1 + \alpha)\text{dist}\,(X, \partial\Omega)\}.$$

We remark that the dependence of w^* on α is well behaved. In particular, nontangential maximal functions defined using different values of α have comparable L^p-norms (see [15] for example). Throughout this paper, boundary values will be taken in the following sense: We say that $u = f$ on $\partial\Omega$ if for a.e. $P \in \partial\Omega$,

$$\lim_{\substack{X \to P \\ X \in \Gamma_\alpha(P)}} u(X) = f(P).$$

Again, except for a set of measure zero, the existence of this limit is independent of the aperture α. By the outer normal derivative, we mean $\nu(P) \cdot \nabla u(P)$ where the restriction of ∇u to $\partial\Omega$ is defined as above.

Finally, we introduce notation for function spaces. We use $L^2(N)$ for the space of square integrable functions (with respect to surface measure) on N. We let $L^{2,1}(D)$ denote the space of functions on D with one derivative in L^2. More precisely, we say that $f \in L^{2,1}(D)$ if $f(X', \phi(X'))$ is in $L^{2,1}(\{X' : \ X_1 < \psi(X'')\})$.

If u is some extension of f to Ω, we let

$$\nabla_{tan} f = \nabla u - \nu \cdot \nabla u \, \nu$$

denote the tangential gradient of f. Of course, this quantity is independent of the particular extension u. With this notation, we will use the norm

$$\|f\|^2_{L^{2,1}(\partial\Omega)} = \int_{\partial\Omega} f^2 + |\nabla_{tan} f|^2 \, dP.$$

Our main result for graph domains is

Theorem 1.5. *Let Ω be a graph domain and suppose that Ω, N and D are as in (1.1)-(1.4). Let $f \in L^{2,1}(D)$ and $g \in L^2(N)$, then for each $k \in \mathbf{R} \setminus \{0\}$, there exists a unique solution to (MP) and the solution u satisfies*

$$\int_{\partial\Omega} (\nabla u^*)^2 + k^2 (u^*)^2 \, dP \leq C \left(\int_D k^2 f^2 + |\nabla_{tan} f|^2 \, dP + \int_N g^2 \, dP \right) \tag{1.6}$$

The constant in this estimate depends only on δ_N, δ_D, $\|\nabla \phi\|_\infty$, $\|\nabla \psi\|_\infty$ and the cone opening α. In particular, it is independent of k.

Our main estimate for the mixed problem is contained in the following Lemma.

Lemma 1.7. *Suppose that Ω, N and D are as in (1.1)-(1.4) and that $k \in \mathbf{R} \setminus \{0\}$. If $\Delta u - k^2 u = 0$, $(\nabla u)^* + u^* \in L^2(\partial\Omega)$, then we have*

$$\int_\Omega k^2 u^2 + |\nabla u|^2 \, dP \leq C \left(\int_D k^2 u^2 + |\nabla_{tan} u|^2 \, dP + \int_N \partial_\nu u^2 \, dP \right)$$

where the constant depends on δ_N, δ_D, $\|\nabla \phi\|_\infty$ and $\|\nabla \psi\|_\infty$.

Proof. We write down three identities.

$$\int_\Omega k^2 u^2 + |\nabla u|^2 \, dX = \int_{\partial\Omega} u \partial_\nu u \, dP \tag{1.8}$$

$$\int_{\partial\Omega} u^2 e_n \cdot \nu \, dP = 2 \int_\Omega u \partial_{x_n} u \, dX \tag{1.9}$$

$$\int_{\partial\Omega} |\nabla u|^2 \alpha \cdot \nu - 2 \partial_\nu u \, \alpha \cdot \nabla u \, dP = -2 \int_\Omega k^2 u \, \alpha \cdot \nabla u. \tag{1.10}$$

In (1.10), $\alpha \in \mathbf{R}^n$ is a constant vector. Each of these is proven by a straightforward application of the Gauss divergence theorem. The *a priori* assumption that $u^* + (\nabla u)^* \in L^2(\partial\Omega)$ imply that the boundary terms at infinity vanish. Note also that (1.8) is Green's first identity and the third identity (1.10) is the Rellich identity.

If, in the third identity, we let α be the vector $(1, 0'', (\delta_D - \delta_N)/2)$, then (1.2) implies that

$$\begin{aligned} \alpha \cdot \nu(Q) &\leq -\tfrac{1}{2}(\delta_D + \delta_N)(1 + \|\nabla\phi\|_\infty^2)^{-1/2}, \quad \text{for a.e. } Q \in D \\ \alpha \cdot \nu(Q) &\geq \tfrac{1}{2}(\delta_D + \delta_N)(1 + \|\nabla\phi\|_\infty^2)^{-1/2}, \quad \text{for a.e. } Q \in N. \end{aligned} \tag{1.11}$$

This sign change is crucial in our application of the Rellich identity to (MP). We decompose $\alpha = \alpha_{tan} + \alpha \cdot \nu\, \nu$ into tangential and normal components. Using this decomposition of α and (1.11) in (1.10), adding $\int_N k^2 u^2$ to both sides and rearranging terms gives

$$\int_D |\partial_\nu u|^2 \, dP + \int_N k^2 u^2 + |\nabla_{tan}\, u|^2 \, dP$$

$$\leq C \int_D |\nabla_{tan} u|^2 + 2\alpha_{tan} \cdot \nabla u \, \partial_\nu u \, dP$$

$$+ \int_N (\partial_\nu u)^2 + k^2 u^2 + 2\alpha_{tan} \cdot \nabla u \, \partial_\nu u \, dP - 2\int_\Omega k^2 u \alpha \cdot \nabla u \, dX.$$

We apply Young's inequality ($2ab \leq \epsilon a^2 + \epsilon^{-1} b^2$) to the cross terms $\alpha_{tan} \cdot \nabla u \partial_\nu u$ and absorb the tangential gradient on D and the normal derivative on N into the lefthand side. This gives

$$\begin{aligned} &\int_D |\partial_\nu u|^2 \, dP + \int_N k^2 u^2 + |\nabla_{tan}\, u|^2 \, dP \\ &\quad \leq C \int_D |\nabla_{tan} u|^2 \, dP + \int_N (\partial_\nu u)^2 + k^2 u^2 \, dP \\ &\quad\quad -2 \int_\Omega k^2 u \alpha \cdot \nabla u \, dX. \end{aligned} \tag{1.12}$$

We apply Young's inequality to the volume integral in (1.12) and then (1.8) which gives

$$\begin{aligned} \left| \int_\Omega k^2 u \alpha \cdot \nabla u \, dX \right| &\leq \tfrac{|\alpha|}{2} \int_\Omega |k|^3 u^2 + |k| |\nabla u|^2 \, dX \\ &\leq \tfrac{|\alpha|}{2} \int_{\partial\Omega} |k| u \partial_\nu u \, dP. \end{aligned} \tag{1.13}$$

Also, using that $-e_n \cdot \nu \geq \delta > 0$, (1.9) and (1.8) gives

$$\begin{aligned} \int_{\partial\Omega} k^2 u^2 \, dP &\leq C \int_\Omega k^2 u \partial_{x_n} u \, dX \\ &\leq C' \int_\Omega |k|^3 u^2 + |k| \, |\nabla u|^2 \, dX \\ &= C' \int_{\partial\Omega} |k| u \partial_\nu u \, dP. \end{aligned} \tag{1.14}$$

Substituting the observations (1.13) and (1.14) into (1.12), breaking the integral of $u\partial_\nu u$ into integrals over D and N and using Young's inequality gives

$$\int_D \partial_\nu u^2 \, dP + \int_N k^2u^2 + |\nabla_{tan}\, u|^2 \, dP$$
$$\leq C\left[\int_D |\nabla_{tan}\, u|^2 + \tfrac{k^2}{\epsilon}u^2 + \epsilon\partial_\nu u^2 \, dP + \int_N \left(1+\tfrac{1}{\epsilon}\right)\partial_\nu u^2 + \epsilon k^2u^2 \, dP\right].$$

Choosing ϵ to be small establishes the Lemma. $\Box$

Next, we recall some results for the Dirichlet and Neumann problem which will be useful to us below. When $k = 0$, these results are due to Jerison and Kenig [9] and Verchota [16]. (See also [5] for a treatment of graph domains.) The extension to certain nonzero k is treated in the author's joint work with Shen [2]. The argument is similar to the proof of Lemma 1.7.

Theorem A. *Let Ω be a graph domain $\{X : \ X_n > \phi(X')\}$ defined by a Lipschitz function ϕ and let $k \in \mathbf{R}$. For each $f \in L^{2,1}(\partial\Omega)$, there exists a solution to the Dirichlet problem*

$$\begin{cases} \Delta u - k^2u = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases}$$

and this solution satisfies

$$\int_{\partial\Omega} (\nabla u^*)^2 + k^2(u^*)^2 \, dP \leq C\int_{\partial\Omega} k^2f^2 + |\nabla_{tan}\, f|^2 \, dP.$$

If $g \in L^2(\partial\Omega)$, then the Neumann problem

$$\begin{cases} \Delta u - k^2u = 0, & \text{in } \Omega \\ \partial_\nu u = g, & \text{on } \partial\Omega \end{cases}$$

has a solution which satisfies

$$\int_{\partial\Omega} (\nabla u^*)^2 + k^2(u^*)^2 \, dP \leq C\int_{\partial\Omega} g^2 \, dP.$$

The constants in the above two estimates depend only on $\|\nabla\phi\|_\infty$ and α. Furthermore, these solutions are unique in the class of functions with $(\nabla u)^* \in L^2(\partial\Omega)$ and $u^* \in L^2(\partial\Omega)$. Finally, consider the map $f \to S_Df \equiv \nabla u$ where u solves the Dirichlet problem with data f and the map $g \to S_Ng \equiv \nabla u$ where

u solves the Neumann problem with data g. Then each of these maps is a continuous function of ϕ, the function which defines $\partial\Omega$, in the sense that if $\pi: \partial\Omega \to \mathbf{R}^{n-1}$ is the map $\pi(X', \phi(X')) = X'$, then $\phi \to \pi^{-1} \circ S_D \circ \pi$ and $\phi \to \pi^{-1} \circ S_N \circ \pi$ are continuous from the space of Lipschitz functions with norm $\|\nabla\phi\|_\infty$ into the space of bounded operators from $L^{2,1}(\mathbf{R}^{n-1})$ and $L^2(\mathbf{R}^{n-1})$ (respectively $L^2(\mathbf{R}^{n-1})$ into $L^2(\mathbf{R}^{n-1})$) with the operator norm.

The statement regarding the continuous dependence of the operator appears in [4] where it is attributed to A. McIntosh. The proof of this fact relies on G. Verchota's representation of solutions by potential operators which depend continuously on ϕ. (See Hofmann [8], for a detailed treatment of the continuous dependence of the potential operators on ϕ.)

The next step in the proof of Theorem 1.5 is to consider the special case where one of the faces lies in a hyperplane. Before proceeding we introduce a characterization of rotations of graph domains. A set Ω is of the form $\{X: X\cdot\beta > F(X - X\cdot\beta\,\beta)\}$ for some unit vector $\beta \in \mathbf{R}^n$ and Lipschitz function $F: \mathbf{R}^n \to \mathbf{R}$ if and only if for some $\alpha > 0$ and all $P \in \partial\Omega$, Ω satisfies the cone conditions

$$\{X: 0 < (X-P)\cdot\beta < (1+\alpha)|X-P|\} \subset \Omega$$

$$\{X: 0 < (P-X)\cdot\beta < (1+\alpha)|X-P|\} \subset \bar{\Omega}^c.$$

Lemma 1.15. *Suppose that Ω, N and D are as in (1.1)-(1.4) and that on the set $\{X': X_1 > \psi(X'')\}$, ϕ is the linear function $\phi(X') = \delta_N X_1$. Then for each $k \in \mathbf{R}\setminus\{0\}$, the problem (MP) has a solution and this solution satisfies*

$$\int_{\partial\Omega} (\nabla u^*)^2 + k^2(u^*)^2\,dP \le C\left(\int_N g^2\,dP + \int_D |\nabla_{tan} f|^2 + k^2 f^2\,dP\right).$$

The constant depends on δ_N, δ_D, $\|\nabla\phi\|_\infty$, $\|\nabla\psi\|_\infty$ and α This solution is unique among the class of u with $\nabla u^* + u^* \in L^2(\partial\Omega)$.

Proof. Uniqueness follows from the energy estimate in Lemma 1.7. To establish existence, it suffices to consider the case where the Neumann data, g, is zero. The sufficiency follows from Theorem A.

To solve (MP) with $g = 0$, we introduce the reflection in the hyperplane containing N. Let $R(X) = X - 2e_N\cdot Xe_N$ where $e_N = (-\delta_N, 0'', 1)/\sqrt{1+\delta_N^2}$ is the unit normal to the hyperplane containing N. We let $\tilde{\Omega}$ be the interior of the set $\bar{\Omega} \cup R(\bar{\Omega})$. One can show that $\tilde{\Omega}$ satisfies the characterization of graph domains given before this Lemma with $\beta = (1, 0'', \delta_N)/\sqrt{1+\delta_N^2}$. Using R, we extend the Dirichlet data f to a function defined on $\partial\tilde{\Omega}$ by letting

$$\tilde{f}(P) = \begin{cases} f(P), & P \in D \\ \\ f(R(P)), & R(P) \in D. \end{cases}$$

Let $\tilde{u}$ be the solution to the Dirichlet problem with data $\tilde{f}$ given by Theorem A. Since $\tilde{f} \circ R = \tilde{f}$, it follows that if $\tilde{u}$ is the solution to the Dirichlet problem in $\tilde{\Omega}$ with data $\tilde{f}$, then $\tilde{u} \circ R = \tilde{u}$. Furthermore, we have $e_N \cdot \nabla\tilde{u} = 0$ on N. Hence, if we let $u = \tilde{u}|_\Omega$, then u will solve (MP) with Neumann data $g = 0$ and Dirichlet data $f \in L^{2,1}(D)$. The nontangential maximal function estimates for u on $\partial\Omega$ follow from the corresponding estimates for $\tilde{u}$ on $\partial\tilde{\Omega}$. $\square$

Our next Lemma provides a proof of the existence part of Theorem 1.5.

Lemma 1.16. *Let Ω, N and D be as in (1.1)-(1.4) and let $k \in \mathbf{R}$. Then for each pair $(f,g) \in L^{2,1}(D) \times L^2(N)$, there exists a solution to (MP) which satisfies the estimate (1.6).*

Proof. As in the previous Lemma, we may assume that $g = 0$. We consider the map $A:\ L^2(D) \to L^{2,1}(D)$ given by

$$Ah = u_h|_D$$

where u_h is the solution to the following Neumann problem provided by Theorem A.

$$\begin{cases} \Delta u - k^2 u = 0, & \text{in } \Omega \\ \partial_\nu u = 0, & \text{on } N \\ \partial_\nu u = h, & \text{on } D. \end{cases}$$

Given Theorem A, solving (MP) is equivalent to showing the map A is bijective. Of course, the uniqueness of solutions to the Neumann problem implies that A is injective. In order to show that A is surjective, we let

$$\phi_0(X') = \max\{\delta_N X_1, (\delta_N + \delta_D)\psi(X'') - \delta_D X_1\}$$

and set $\phi_t = t\phi + (1-t)\phi_0$. As before, we set $N_t = \partial\Omega_t \cap \{X : X_1 \geq \psi(X'')\}$ and $D_t = \partial\Omega_t \cap \{X :\ X_1 < \psi(X'')\}$. We let $\pi_t :\ \partial\Omega_t \to \mathbf{R}^{n-1}$ be the projection $\pi_t(X', \phi_t(X')) = X'$. Observe that the family of Ω_t, N_t and D_t, $\ 0 \leq t \leq 1$, satisfy the conditions (1.1)-(1.4) with constants independent of t. In fact, the dividing surface between D_t and N_t projects under π_t to the surface $\{X' : X_1 = \psi(X'')\}$ which is independent of t. We let $\tilde{A}_t : L^2(\pi_t(D_t)) \to L^{2,1}(\pi_t(D_t))$ be the operator given by

$$\tilde{A}_t(h) = [A_t(f \circ \pi_t^{-1})] \circ \pi_t$$

where A_t is the Neumann to Dirichlet map on $\partial\Omega_t$. The surjectivity of $\tilde{A}_1$ and hence $A = A_1$ will follow from the following facts. (See [6, Theorem 5.2]).

1) $\tilde{A}_0$ is surjective. 2) $t \to \tilde{A}_t$ is continuous from $[0,1]$ into bounded operators from $L^2(\pi_t(D_t))$ to $L^{2,1}(\pi_t(D_t))$. 3) $\int_{D_t} k^2(A_t h)^2 + (\nabla_{tan} A_t h)^2 \, dP \geq c\int_{D_t} h^2 \, dP$.

Fact 1) is a restatement of Lemma 1.15, 2) is contained in Theorem A and 3) is a consequence of Lemma 1.7.

According to 3) above, the norm of A^{-1} is determined by the constant in Lemma 1.7. This and Theorem A give the nontangential maximal function estimates of Theorem 1.5. $\square$

Proof of Theorem 1.5. The existence assertion in Theorem 1.5 is in Lemma 1.16. The uniqueness follows from the energy estimate (1.8) in Lemma 1.7. $\square$

§2 BOUNDED DOMAINS.

We now turn our attention to proving existence and regularity of solutions to (MP) in bounded domains. We will use Theorem 1.5 as our main tool. As mentioned in the Introduction, it is an important technical point that we have the estimates of Theorem 1.5 for large k. For these values of k, it is simpler to use a partition of unity to localize the problem since the estimate of $\int_{\partial\Omega} k^2u^2$ in (1.6) can be used to control errors that arise in the localization process.

We begin by giving a precise description of the boundary value problems we will study. Let Ω be a bounded connected open set and assume that N and D form a decomposition of $\partial\Omega$ with D nonempty. We assume that there is a finite collection of points $\{P_i : i = 1, ..., M\}$ and $r > 0$ so that $\partial\Omega \subset \bigcup_{i=1}^{M} B_r(P_i)$ and that for each i, there is a domain Ω_i, with boundary decomposition N_i, D_i, which are orthogonal motions of graph domains satisfying (1.1)-(1.4) and so that

$$\Omega \cap B_{2r}(P_i) = \Omega_i \cap B_{2r}(P_i),$$

$$D \cap B_{2r}(P_i) = D_i \cap B_{2r}(P_i),$$

$$N \cap B_{2r}(P_i) = N_i \cap B_{2r}(P_i).$$

Our main result is:

Theorem 2.1. *Let Ω, N and D be as described above. Then the problem (MP) (with $k = 0$) has a solution which satisfies*

$$\int_{\partial\Omega} (\nabla u^*)^2 \, dP \leq C \left(\int_N g^2 \, dP + \int_D |\nabla_{tan} f|^2 + f^2 \, dP \right).$$

The constant C is determined by geometric properties of the domain. (See the remarks following (2.12).) This solution is unique among the class of harmonic functions with $\nabla u^* \in L^2(\partial\Omega)$.

Our first Lemma will be used in conjunction with the Fredholm theory to reduce the solution of (MP) with $k = 0$ to the solution of (MP) with k large. This lemma requires only that Ω be a bounded Lipschitz domain. This means that there is a finite collection of orthogonal motions of graph domains $\{\Omega_i : \ i = 1, ..., m\}$ and points $P_i \in \partial\Omega$ so that

$$\Omega \cap B_{2r}(P_i) = \Omega_i \cap B_{2r}(P_i)$$

$$\partial\Omega \cap B_{2r}(P_i) = \partial\Omega_i \cap B_{2r}(P_i).$$

Lemma 2.2. *Let Ω be a bounded Lipschitz domain for $k \in \mathbf{R}$, consider the map $S_k : \ L^{2,1}(\partial\Omega) \to L^2(\partial\Omega)$ given by $S_k f = \partial_\nu u$ where u is the solution of*

$$\begin{cases} \Delta u - k^2 u = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega. \end{cases}$$

For each pair of real numbers, k_1 and k_2, the map $S_{k_1} - S_{k_2}$ is compact.

Proof. Let $f \in L^{2,1}(\partial\Omega)$ and let u_i be the solution of $(\Delta - k_i^2)u_i = 0$ with Dirichlet data f. According to Theorem A (extended to bounded domains), we have $\nabla u_i^* \in L^2(\partial\Omega)$. If we let $w = u_1 - u_2$, then the Rellich identity gives the estimate

$$\int_{\partial\Omega} |\partial_\nu w|^2 \, dP \le C \left[\int_\Omega |\nabla w| \, |k_1^2 u_1 - k_2^2 u_2| + |\nabla w|^2 \, dX \right]. \tag{2.3}$$

(See [2, 9]).

The nontangential maximal function estimates for ∇w imply that

$$\int_{\Omega_\epsilon} |\nabla w|^2 \, dX \le \epsilon C_{k_1,k_2} \int_{\partial\Omega} |\nabla f|^2 + f^2 \, dP$$

where $\Omega_\epsilon = \{X \in \Omega : \mathrm{dist}(X, \partial\Omega) < \epsilon\}$. This and easy interior compactness properties of solutions of $(\Delta - k_i^2)u_i = 0$ imply that the map $f \to \nabla w \equiv Af$ is a compact map from $L^{2,1}(\partial\Omega) \to L^2(\Omega)$. Hence, we may rewrite (2.3) as

$$\|S_{k_1} f - S_{k_2} f\|^2_{L^2(\partial\Omega)} \le C_{k_1,k_2}(\|Af\|_{L^2(\Omega)} \|f\|_{L^2(\partial\Omega)} + \|Af\|^2_{L^2(\partial\Omega)}).$$

This and the compactness of A imply the Lemma. $\square$

We are now ready for the proof of Theorem 2.1.

Proof of Theorem 2.1. We let ξ_i, $i = 1, ..., M$ be a smooth partition of unity on $\partial\Omega$ which is subordinate to the cover $B_r(P_i)$. For $i = 1, .., M$ we let v_i be the solution to (MP) in Ω_i with data $\xi_i f$ on D_i and $\xi_i g$ on N_i. For each i, we let η_i be a smooth cutoff function which is one on $B_r(P_i)$ and is supported in $B_{2r}(P_i)$. We let $u_1 = \sum_{i=1}^{M} \eta_i v_i$ and observe that $u_1 - f = 0$ on D and on N, $\partial_\nu u_1 - g = \sum_{i=1}^{M} v_i \partial_\nu \eta_i$. Hence,

$$\begin{aligned} \int_N |\partial_\nu u_1 - g|^2 \, dP &\le C \sum_i \int_{N_i} v_i^2 \, dP \\ &\le \tfrac{C}{k^2} \left[\int_N g^2 \, dP + \int_D (1 + k^2) f^2 + |\nabla_{tan} f|^2 \, dP \right]. \end{aligned} \tag{2.4}$$

Where we have used the estimate of Theorem 1.5 to bound the $L^2(N_i)$-norm of v_i by the data of v_i on $\partial\Omega_i$. Thus for large k, the data of u_1 closely approximates the desired data f and g. Unfortunately,

$$\Delta u_1 - k^2 u_1 = \sum_i (v_i \Delta \eta_i + 2\nabla \eta_i \cdot \nabla v_i) \equiv F \ne 0.$$

We let E_k be the fundamental solution for $\Delta - k^2$ and let $u_2 = E_k * F$. We claim that

$$\begin{aligned}\int_{\partial\Omega} k^2 u_2^2 + |\nabla u_2|^2 &\leq \tfrac{C}{k^2}\left[\int_N g^2\, dP\right.\\ &\qquad \left. + \int_D |\nabla_{tan} f|^2 + k^2 f^2\, dP\right], \quad |k| \geq 1.\end{aligned} \tag{2.5}$$

To establish the claim (2.5), we begin by observing that

$$|k| \int_{\Omega_i} k^2 v_i^2 + |\nabla v_i|^2\, dP \leq C\left[\int_{D_i} k^2 f_i^2 + |\nabla_{tan} f_i|^2\, dP + \int_{N_i} g_i^2\, dP\right]. \tag{2.6}$$

which follows from the energy inequality (1.8), Young's inequality ($2ab \leq a^2 + b^2$) and the estimate of Lemma 1.7. Next observe that for multiindices α, with $|\alpha| \leq 2$, the map $h \to \partial_X^\alpha E_k * h$ is bounded on $L^2(\mathbf{R}^n)$ with norm $C|k|^{2-|\alpha|}$

$$\|\partial_x^\alpha E_k * h\|_{L^2(\mathbf{R}^n)} \leq C|k|^{|\alpha|-2}\|h\|_{L^2(\mathbf{R}^n)}.$$

Hence, we have the estimate

$$\begin{aligned}&\int_{\mathbf{R}^n} k^4 u_2^2 + k^2|\nabla u_2|^2 + |\nabla^2 u_2|^2\, dX\\ &\qquad \leq C\sum_i \int_{\Omega_i} v_i^2 + |\nabla v_i|^2.\end{aligned} \tag{2.7}$$

Next, observe that if $\alpha : \bar{\Omega} \to \mathbf{R}^n$ is C^1, then

$$\int_{\partial\Omega} u_2^2 \alpha \cdot \nu\, dP = \int_\Omega 2\alpha \cdot \nabla u_2 u_2 + u_2^2 \operatorname{div} \alpha\, dX. \tag{2.8}$$

On a Lipschitz domain, we may choose α so that $\alpha \cdot \nu \geq \delta > 0$ a.e. on $\partial\Omega$. Thus using Young's inequality and (2.7) followed by (2.6) we obtain

$$\begin{aligned}\int_{\partial\Omega} k^2 u_2^2\, dP &\leq C|k| \int_\Omega |\nabla u_2|^2 + k^2 u_2^2\, dX\\ &\leq \tfrac{C}{|k|} \cdot \sum_{i=1}^M \int_{\Omega_i} v_i^2 + |\nabla v_i|^2\, dX\\ &\leq \tfrac{C}{k^2} \sum_{i=1}^M \left(\int_{N_i} g_i^2\, dP + \int_{D_i} |\nabla_{tan} f_i|^2 + k^2 f_i^2\, dP\right).\end{aligned}$$

A similar argument gives the same bound for $\int_{\partial\Omega} |\nabla u_2|^2\, dP$. This establishes the claim (2.5).

Now let $u = u_1 - u_2$. We have $(\Delta - k^2)u = 0$ and the estimates (2.4) and (2.5) imply that

$$\begin{aligned} &\int_N (\partial_\nu u - g)^2 \, dP + \int_D |\nabla_{tan} u - f|^2 + k^2(u-f)^2 \, dP \\ &\qquad \leq \tfrac{C}{k^2} \left[\int_N g^2 \, dP + \int_D k^2 f + |\nabla_{tan}(u-f)|^2 \, dP \right], \quad |k| \geq 1. \end{aligned} \tag{2.9}$$

Finally, we claim that $(\nabla u)^*$ and u^* are in $L^2(\partial\Omega)$. This is true for u_1 by Theorem 1.5. For u_2, we have $\nabla^2 u_2 \in L^2(\Omega)$ and this implies that ∇u_2^* and u_2^* are in $L^2(\partial\Omega)$. (See [2, Lemma 1.2]). The estimate (2.9) implies that the map $(f, g) \to (u|_D, \partial_\nu u|_N)$ is invertible on the space $L^{2,1}(D) \times L^2(N)$, when k is large. Thus we may solve (MP) in bounded domains for k sufficiently large.

To remove the restriction that k is large, we consider the map $\Lambda_k : \ L^{2,1}(N) \to L^2(N)$ given by $\Lambda_k f = \partial_\nu u|_N$ where u is the solution to the Dirichlet problem

$$\begin{cases} (\Delta - k^2)u = 0, & \text{in } \Omega \\ u = f, & \text{on } N \\ u = 0, & \text{on } D. \end{cases}$$

We write $\Lambda_0 = \Lambda_k + \Lambda_0 - \Lambda_k$ for some k large and will show that Λ_0 is invertible. Note first that the uniqueness of solutions to (MP) implies that Λ_0 is injective. Next, observe that the solvability of (MP) for large k which has established in the previous paragraph implies that the map Λ_k is invertible and thus of index zero. Lemma 2.2 asserts that $\Lambda_k - \Lambda_0$ is compact and hence Λ_0 is also of index zero. Thus Λ_0 is invertible and we may solve (MP) when $k = 0$. Of course, this argument will also work for other values of k besides 0.

This soft argument does not give good estimates for the nontangential maximal function of the solution. We will show that such estimates do in fact hold. For $k = 0$, we have

$$\int_{\partial\Omega} (\nabla u^*)^2 \, dP \leq C \left[\int_D f^2 + |\nabla_{tan} f|^2 \, dP + \int_N g^2 \, dP \right]$$

where the constant depends on explicit geometric quantities. To see this, observe that we can construct a smooth vector field $\beta : \ \bar{\Omega} \to \mathbf{R}^n$ which satisfies

$$\beta(Q) \cdot \nu(Q) \geq \delta > 0 \;\text{ on } D$$

and

$$\beta(Q) \cdot \nu(Q) \leq -\delta < 0 \;\text{ on } N.$$

This is done by choosing the constant vector used in (1.10) for each graph domain Ω_i and then patching together with a partition of unity. Using the vector field β in the Rellich identity and (2.8) in place of (1.9) we can mimic the proof of Lemma 1.7 in Ω, and show that if $\Delta u = 0$, $(\nabla u)^* \in L^2(\partial\Omega)$, then

$$\int_{\partial\Omega} |\nabla u|^2 \, dP \le C \left[\int_D |\nabla_{tan} u|^2 \, dP + \int_N \partial_\nu u^2 \, dP + \int_\Omega |\nabla u|^2 \, dX \right]. \tag{2.10}$$

The energy estimate (1.8) gives

$$\begin{aligned} 2\int_\Omega |\nabla u|^2 \, dX = 2\int_\Omega u \partial_\nu u \, dP \le \int_D \tfrac{1}{\epsilon} u^2 + \epsilon (\partial_\nu u)^2 \, dP \\ + \int_N \tfrac{1}{\epsilon} (\partial_\nu u)^2 + \epsilon u^2 \, dP \end{aligned} \tag{2.11}$$

Since

$$\int_N u^2 \, dP \le C \left(\int_D u^2 \, dP + \int_\Omega |\nabla u|^2 \, dX \right) \tag{2.12}$$

when Ω is connected, we may conclude from (2.10), (2.11) and (2.12) that

$$\int_{\partial\Omega} |\nabla u|^2 \le C \left[\int_D u^2 + |\nabla_{tan} u|^2 \, dP + \int_N \partial_\nu u^2 \, dP \right].$$

The constant in this estimate depends only on the vector fields α and β and the constant in the Poincaré inequality (2.12). Finally, the estimates of Dahlberg [9] give that

$$\int_{\partial\Omega} (\nabla u^*)^2 \, dP \le C \int_{\partial\Omega} |\nabla u|^2 \, dP$$

where the constant depends only on the Lipschitz character of Ω. $\Box$

The result of Theorem 2.1 may be viewed as a regularity result for the weak solution of (MP) constructed using energy estimates: If the boundary data (f, g) lie in $L^{2,1}(D) \times L^2(N)$, then the solution u lies not only in the energy space, $L^{2,1}(\Omega)$, but also $\nabla u \in L^{2,1}(\partial\Omega)$. We give a simple example which shows that this regularity may fail in some Lipschitz domains.

Examples. Let $0 < \alpha < 2\pi$ and consider the sector $S_\alpha = \{z : \ 0 < \arg z < \alpha\}$. The function $r^{2\pi/\alpha} \sin(\pi\theta/2\alpha)$ is harmonic in this sector, satisfies $\partial_\nu u = 0$ on $\{re^{i\alpha} : \ r > 0\}$ and $u = 0$ on $\{z : \ \mathrm{Im}\ z = 0, \ \mathrm{Re}\ z > 0\}$. If we let $\Omega_\alpha = S_\alpha \cap \{|z| < 1\}$, $N_\alpha = \{re^{i\alpha} : \ 0 \le r \le 1\}$ and $D_\alpha = \partial\Omega_\alpha \setminus N_\alpha$, then $\Omega_\alpha, N_\alpha, D_\alpha$ satisfy the hypotheses of Theorem 2.1, except near $z = 0$ when $2\pi > \alpha \ge \pi$. If we let $u_\alpha = r^{\pi/2\alpha} \sin(\theta\pi/2\alpha)$ on Ω_α, then $\partial_\nu u_\alpha \in L^2(N_\alpha)$, $u_\alpha \in L^2(D_\alpha)$

but if $\alpha > \pi$, then $\nabla u_\alpha \notin L^2(\partial\Omega_\alpha)$. Yet we still have $\nabla u_\alpha \in L^2(\Omega_\alpha)$ for all α, hence u_α is the unique solution of (MP) in $L^{2,1}(\Omega_\alpha)$. Thus the conclusion of Theorem 2.1 fails in the family of domains Ω_α precisely when Ω_α violates (1.2).

Another remark indicating the correctness of the class of domains considered in Theorem 2.1 is an elementary observation of C. Kenig. Suppose that Theorem 2.1 holds for (MP) on a given Ω with N and D a decomposition of $\partial\Omega$. If u is the solution of (MP), with data f on D and data 0 on N, then the map $f \to u|_{\partial\Omega}$ provides an extension operator from $L^{2,1}(D)$ into $L^{2,1}(\partial\Omega)$. It is well-known that the existence of an extension operator requires some conditions on D. One well-known sufficient condition for the existence of an extension operator for domains in $\mathbf{R}^n$ is that the domain be Lipschitz.

We also mention several questions raised by Theorem 2.1. 1) Is it possible to obtain similar results when $f \in L^{p,1}(D)$ and $g \in L^p(N)$? The examples given above indicate that for $p < 2$, one should be able to obtain regularity results in domains with nonconvex or "re-entrant corners". 2) Can one study (MP) in domains where the decomposition of $\partial\Omega$ into D and N is more complicated? For example, what can be said about (MP) in the pyramid $|X_1| + |X_2| < X_3$ if we specify Dirichlet and Neumann data on alternate faces.

References

[1] Russell M. Brown, *The method of layer potentials for the heat equation in Lipschitz cylinders*, Amer. J. Math. **111** (1989), 339–379.

[2] Russell M. Brown and Z. Shen, *The initial-Dirichlet problem for a fourth-order parabolic equation in Lipschitz cylinders*, Ind. Univ. Math. J. **39** (1990), 1313–1353.

[3] Jindrich Neças, *Les methodes directes en theorie des equations elliptique*, Masson et Cie., Editeurs, Paris, 1967, Translated from Czech.

[4] B.E.J. Dahlberg and C.E. Kenig, *Harmonic analysis and partial differential equations*, Tech. Report 1985-03, Chalmers Institute of Technology and The University of Göteborg, 1985.

[5] ______, *Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains*, Ann. of Math. **125** (1987), 437–466.

[6] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.

[7] Pierre Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, 1985.

[8] S. Hofmann, *On singular integrals of Caldéron-type in R^n, and BMO*, To appear, Rev. Mat. Iberoamericana.

[9] D.S. Jerison and C.E. Kenig, *Boundary value problems in Lipschitz domains*, Studies in partial differential equations (Washington, D.C.) (Walter Littman, ed.), MAA Studies in Mathematics, vol. 23, Math. Assoc. Amer., Washington, D.C., 1982, pp. 1–68.

[10] ———, *The Neumann problem on Lipschitz domains*, Bull. Amer. Math. Soc. **4** (1982), 203–207.

[11] V.A. Kondrachev, *Boundary value problmes for elliptic equaitons in domains with conical or angular points*, Transactions Moscow Math. Soc. **16** (1967), 227–313.

[12] L. Payne and H. Weinberger, *New bounds for solutions of second order elliptic partial differential equations*, Pac. J. Math. **8** (1958), 551–573.

[13] Z. Shen, *Layer potentials and boundary value problems for parabolic Lamé systems of elasticity and a nonstationary linearized system of Navier-Stokes equations in Lipschitz cylinders*, Ph.D. thesis, University of Chicago, 1989.

[14] ———, *Boundary value problems for parabolic Lamé systems and a nonstationary linearized system of Navier-Stokes equations in Lipschitz cylinders*, Amer. J. Math. **113** (1991), 293–373.

[15] A. Torchinsky, *Real variable methods in harmonic analysis*, Academic Press, 1986.

[16] G.C. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation on Lipschitz domains*, J. Funct. Anal. **59** (1984), 572–611.