The mixed problem in L^p for some two-dimensional Lipschitz domains

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Abstract

We consider the mixed problem,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = f_N & \text{on N} \\ u = f_D & \text{on } D \end{cases}$$

in a class of Lipschitz graph domains in two dimensions with Lipschitz constant at most 1. We suppose the Dirichlet data, f_D, has one derivative in $L^p(D)$ of the boundary and the Neumann data, f_N, is in $L^p(N)$. We find a $p_0 > 1$ so that for p in an interval $(1, p_0)$, we may find a unique solution to the mixed problem and the gradient of the solution lies in L^p.

1 Introduction

The goal of this paper is to study the mixed problem (or Zaremba's problem) for Laplace's equation in certain two-dimensional domains when the Neumann data

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comes from L^p and the Dirichlet problem has one derivative in L^p. We will consider both L^p with respect to arc-length, $d\sigma$ and also $L^p(w\,d\sigma)$ where the weight w is of the form $w(x) = |x|^\epsilon$. By the *mixed problem for* $L^p(w\,d\sigma)$, we mean the following boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f_D & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N \\ (\nabla u)^* \in L^p(w\,d\sigma) & \end{cases} \tag{1.1}$$

Here, $(\nabla u)^*$ is the non-tangential maximal function of the gradient (see the definition in (2.1)). The domain Ω will be a Lipschitz graph domain. Thus, $\Omega = \{(x_1, x_2) : x_2 > \phi(x_1)\}$ where $\phi : \mathbf{R} \to \mathbf{R}$ is a Lipschitz function with $\phi(0) = 0$. We will call such domains *standard Lipschitz graph domains.* The sets D and N satisfy $D \cup N = \partial\Omega$ and $D \cap N = \emptyset$. Also, we want D to be open, so that it supports Sobolev spaces. Here, we will generally assume that f_D is a function with one derivative in $L^p(D, w\,d\sigma)$ and that f_N is in $L^p(N, w\,d\sigma)$. Our goal is to find conditions on the data, the exponent p and the weight w so that the problem (1.1) has a unique solution. This paper continues the work of Sykes and Brown [4, 33, 34] to provide a partial answer to problem 3.2.15 from Kenig's CBMS lecture notes [20]. Our goal is to obtain L^p-results in a class of domains that is not included in the domains studied by Sykes [33].

We recall a few works which study the regularity of the mixed problem. In [1], Azzam and Kreyszig established that the mixed problem has a solution in $C^{2+\alpha}$ in bounded two-dimensional domains provided that N and D meet at a sufficiently small angle. Lieberman [23] also gives conditions which imply that the solution is Hölder continuous. In addition, much effort has been devoted to problems in polygonal domains, see the monograph of Grisvard [16]. Savaré [29] finds solutions in the Besov space $B^{3/2}_{2,\infty}$ in smooth domains. This positive result fits quite nicely with the example below which shows that there is a solution whose gradient just misses having non-tangential maximal function in $L^2(d\sigma)$. It is known that if a function is harmonic and the non-tangential maximal function of the gradient is in $L^2(d\sigma)$, then the function also belong to the Besov space $B^{3/2}_{2,2}$ (see the article of Fabes [12]). Recent work of I. Mitrea and M. Mitrea [27] extend the class of spaces where we know the mixed problem is well-posed. However, their work does not extend the class of domains beyonds those considered in [4]. Recent work of Mazya and Rossman [24, 26, 25] consider mixed boundary value problems for the Stokes system and the Navier Stokes system in polyhedral domains. It would be of interest to extend the methods presented here to other equations.

If we recall the standard tools for studying boundary value problems in Lipschitz domains, we see that the mixed problem presents an interesting technical challenge. On the one hand, the starting point for many results on boundary value problems in Lipschitz domains is the Rellich identity (see Jerison and Kenig [18], for example). This remarkable identity provides estimates at the boundary for derivatives of a harmonic function in L^2. On the other hand, simple and well-known examples show that the mixed problem is not solvable in $L^2(d\sigma)$ for smooth domains. Let us recall an example in the upper half-space, $\mathbf{R}^2_+ = \{(x_1, x_2) : x_2 > 0\}$. We let $N = \{(x_1, 0),\ x_1 > 0\}$, $D = \{(x_1, 0),\ x_1 < 0\}$, and we consider the harmonic function

$$u(z) = Re\ (\sqrt{z} - \sqrt{z+i}), \qquad z = x_1 + ix_2.$$

It is easy to see that we have

$$\begin{aligned} \left|\frac{\partial u}{\partial \nu}\right| &\leq C(1+|z|)^{-3/2} \quad \text{on} \quad N, \\ \left|\frac{du}{d\sigma}\right| &\leq C(1+|z|)^{-3/2} \quad \text{on} \quad D, \end{aligned}$$

but

$$|\nabla u| \geq \frac{C}{\sqrt{|z|}} \qquad \text{for} \quad |z| \leq 1/2.$$

Hence, we do not have $(\nabla u)^* \in L^2_{loc}(\mathbf{R}, d\sigma)$. On the other hand we have $(\nabla u)^* \in L^p_{loc}(\mathbf{R}, d\sigma)$ for all $1 \leq p < 2$.

We detour around this problem by establishing weighted estimates in L^2 using the Rellich identity. This relies on an observation of Luis Escauriaza [11] that a Rellich identity holds when the components of the vector field are, respectively, the real and imaginary part of a holomorphic function. Then, we imitate the arguments of Dahlberg and Kenig [10] to establish Hardy-space estimates (with a different weight). The weights are chosen so that interpolation will give us unweighted L^p as an intermediate space. The weights we consider will be of the form $|x|^\epsilon$ restricted to the boundary of a Lipschitz graph domain. Earlier work of Shen [30] (see also §3) gives a different approach to the study of weighted estimates for the Neumann and regularity problems when the weight is a power.

In the result below and throughout this paper, we assume that Ω is a standard Lipschitz graph domain and that

$$D = \{(x_1, \phi(x_1)) : x_1 < 0\}, \qquad N = \{(x_1, \phi(x_1)) : x_1 \geq 0\}. \tag{1.2}$$

We will call Ω with N and D as defined above, a *standard Lipschitz graph domain for the mixed problem.* The *Lipschitz constant* of the domain, M is defined by

$$M = \|\phi'\|_{L^\infty(\mathbf{R})}. \tag{1.3}$$

Our main result is the following.

Theorem 1.1 *Let Ω be a standard Lipschitz graph domain for the mixed problem, with Lipschitz constant M less than 1. There exists $p_0 = p_0(M)$ so that for $1 < p < p_0$,* *if $f_N \in L^p(N, d\sigma)$ and $df_D/d\sigma \in L^p(D, d\sigma)$, then the mixed problem for $L^p(d\sigma)$ has a unique solution. The solution satisfies*

$$\|(\nabla u)^*\|_{L^p(d\sigma)} \leq C(p, M)\left(\|f_N\|_{L^p(N,d\sigma)} + \left\|\frac{df_D}{d\sigma}\right\|_{L^p(D,d\sigma)}\right). \tag{1.4}$$

2 Preliminaries

In this section, we prove uniqueness for the weighted L^p-Neumann and regularity problems in a Lipschitz graph domain, see (2.4) and (2.5). Here and in the sequel, we let ν denote the outer unit normal to $\partial\Omega$. We recall that a *bounded* Lipschitz domain is a bounded domain whose boundary is parameterized by (finitely many) Lipschitz graphs. We begin our development with an observation that we learned from Luis Escauriaza [11].

Lemma 2.1 (L. Escauriaza) *Suppose Ω is a bounded Lipschitz domain and u and $\alpha = (\alpha_1, \alpha_2)$ are smooth in a neighborhood of $\bar{\Omega}$ with $\Delta u = 0$ and $\alpha_1 + i\alpha_2$ holomorphic. Then, we have*

$$\int_{\partial\Omega} |\nabla u|^2 \alpha \cdot \nu - 2\frac{\partial u}{\partial \alpha}\frac{\partial u}{\partial \nu}\, d\sigma = 0.$$

Proof. A calculation shows that $\operatorname{div}(|\nabla u|^2\alpha - (2\alpha \cdot \nabla u)\nabla u) = 0$. Thus, the lemma is an immediate consequence of the divergence theorem. ∎

Remark. In the plane, it is easy to see that $\operatorname{div}(|\nabla u|^2\alpha - 2\nabla u \alpha \cdot \nabla u) = 0$ for all harmonic functions u if and only if the vector field α is holomorphic. In fact, we may replace harmonic by linear in the previous sentence.

Next, we recall Carleson measures and the fundamental property of these measures. The applications we have in mind are simple, but the use of Carleson measures will allow us to appeal to a well-known geometrical argument rather than invent our own.

Let σ be a measure on the boundary of Ω. A measure μ on Ω is said to be a *Carleson measure with respect to σ* if there is a constant A so that for each $x \in \partial\Omega$ and each $r > 0$, we have

$$\mu(B_r(x) \cap \Omega) \leq A\sigma(\Delta_r(x)).$$

Here, we are using $B_r(x)$ to denote the ball (or disc) in $\mathbf{R}^2$ with center x and radius r and we also use $\Delta_r(x) = B_r(x) \cap \partial\Omega$ to denote a ball on the boundary of Ω.

Before we can state the next result, we need a few definitions. For $\theta_0 > 0$, we define $\Gamma(0) = \{re^{i\theta} : r > 0,\ |\theta - \pi/2| < \theta_0\}$ to be the sector with vertex at 0, vertical axis and opening $2\theta_0$. Then, we put $\Gamma(x) = x + \Gamma(0)$, $x \in \partial\Omega$. If Ω is a Lipschitz graph domain with constant M, then we have that $\Gamma(x)$ defines a non-tangential approach region provided $\theta_0 < \pi/2 - \tan^{-1}(M)$. We fix such a θ_0 and if v is a function defined on Ω, we define the *non-tangential maximal function of* v, v^* on $\partial\Omega$ by

$$v^*(x) = \sup_{y \in \Gamma(x)} |v(y)| \,, \quad x \in \partial\Omega \,. \tag{2.1}$$

We also use these sectors to define restrictions to the boundary in the sense of non-tangential limits. For a function v in Ω, we define the restriction of v to the boundary by

$$v(x) = \lim_{\Gamma(x) \ni y \to x} v(y) \,, \quad x \in \partial\Omega \tag{2.2}$$

provided the limit exists. Finally, we recall that a measure σ is a *doubling measure* if there is a constant C so that for all $r > 0$ we have $\sigma(\Delta_{2r}(x)) \leq C\sigma(\Delta_r(x))$.

Proposition 2.2 *If τ is a doubling measure on $\partial\Omega$ and μ is a Carleson measure with respect to τ with constant A, then there exists a constant $C > 0$ so that*

$$\left| \int_\Omega v \, d\mu \right| \leq CA \int_{\partial\Omega} v^* \, d\tau.$$

This result is well-known. The proof in Stein [31, pp. 58–60] easily generalizes from Lebesgue measure to doubling measures.

A simple example of a Carleson measure that will be useful to us is the following. With $R > 0$ and $\epsilon > -1$, define $d\sigma_\epsilon$ and $d\mu_\epsilon$ by

$$d\sigma_\epsilon(x) = |x|^\epsilon d\sigma(x) \,, \ x \in \partial\Omega \quad ; \quad d\mu_\epsilon(y) = \frac{|y|^\epsilon}{R} \chi_{B_R(0)}(y) dy \,, \ y \in \Omega, \tag{2.3}$$

where we are using $d\sigma$ for arc-length measure on $\partial\Omega$ and dy for area measure in the plane. It is not hard to see that σ_ϵ is a doubling measure on $\partial\Omega$ (see Lemma 3.1) and μ_ϵ is a Carleson measure with respect to σ_ϵ.

Lemma 2.3 (Rellich Identity) *Let Ω be a standard Lipschitz graph domain. Given $\epsilon > -1$ and $a \in \mathbf{C}$, we define $\alpha(z) = (\mathrm{Re}(az^\epsilon), \mathrm{Im}(az^\epsilon))$. Here, $z = x_1 + ix_2$. Let σ_ϵ be as in (2.3). If u is harmonic in Ω and $(\nabla u)^* \in L^2(\sigma_\epsilon)$, then we have*

$$\int_{\partial\Omega} |\nabla u|^2 \alpha \cdot \nu - 2\alpha \cdot \nabla u \frac{\partial u}{\partial \nu} \, d\sigma = 0.$$

Remark. Here and in the sequel, we let $\nabla u(x)$ denote the non-tangential limit of ∇u at $x \in \partial\Omega$, see (2.2). It is well known that, with the assumptions of Lemma 2.3, the non-tangential limit of ∇u exists a.e. $x \in \partial\Omega$, see Dahlberg [9] or Jerison and Kenig [17].

Proof. We introduce a cut-off function η_R where $\eta_R(y) = 1$ if $y \in B_R(0)$, $\eta_R(y) = 0$ if $|y| > 2R$, and $|\nabla \eta_R| \le C/R$. For $\tau > 0$, we define a translate of u, u_τ, by $u_\tau(y) = u(y + \tau e_2)$, where $e_2 = (0, 1)$. Note that u_τ is smooth in a neighborhood of $\overline{\Omega}$. Since Ω is a graph domain and $0 \in \partial\Omega$ it follows that az^ϵ is holomorphic in Ω. Thus, we may apply Lemma 2.1 to u_τ and α. The divergence theorem now yields

$$\begin{aligned} \int_{\partial\Omega} \left(|\nabla u_\tau|^2 \alpha \cdot \nu - 2\alpha \cdot \nabla u_\tau \frac{\partial u_\tau}{\partial \nu} \right) \eta_R \, d\sigma &= \int_\Omega \nabla \eta_R \cdot \left(|\nabla u_\tau|^2 \alpha - (2\alpha \cdot \nabla u_\tau) \nabla u_\tau \right) \, dy \\ &\equiv I_R. \end{aligned}$$

The hypothesis $\epsilon > -1$ is needed to justify integration by parts with the singular vector field α. Now, using the measures defined in (2.3), Proposition 2.2 and the definition of α we have

$$|I_R| \le C \int_{\partial\Omega} f_R(x) \, d\sigma_\epsilon$$

where f_R is given by

$$f_R(x) = \sup_{y \in \Gamma(x), |y| > R} |\nabla u_\tau(y)|^2.$$

Notice that for each x, $\lim_{R\to\infty} f_R(x) = 0$. Hence, our assumption that $(\nabla u)^*$ is in $L^2(d\sigma_\epsilon)$ and the Lebesgue dominated convergence theorem imply that $\lim_{R\to\infty} I_R = 0$.

We may now let $\tau \to 0^+$ and use the dominated convergence theorem again to obtain the lemma. ∎

As a step towards studying the mixed problem in weighted spaces, we consider the Neumann and regularity problems in two-dimensions. By the *Neumann problem for $L^p(w \, d\sigma)$*, we mean the problem of finding a function u which satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = f_N, & \text{on } \partial\Omega \\ (\nabla u)^* \in L^p(w \, d\sigma) \end{cases} \tag{2.4}$$

where, in general, we assume that f_N is taken from $L^p(w \, d\sigma)$. We also study the *regularity problem for $L^p(w \, d\sigma)$* where we look for a function u which satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f_D, & \text{on } \partial\Omega \\ (\nabla u)^* \in L^p(w \, d\sigma) \end{cases} \tag{2.5}$$

where, in general, we assume that $df_D/d\sigma$ is in $L^p(w \, d\sigma)$. 2006-

An important component in establishing uniqueness is the following local regular- 06-06,
ity result for solutions with zero data. deletion

Lemma 2.4 *Suppose that $\Delta u = 0$ in Ω, $\frac{\partial u}{\partial \nu} = 0$ or $u = 0$ on $\partial\Omega$ and $(\nabla u)^* \in L^1_{loc}(\partial\Omega)$. Then, for every bounded set $B \subset \Omega$, we have $\nabla u \in L^2(B)$.*

Proof. We consider the case of Neumann boundary conditions. Dirichlet boundary conditions may be handled by a similar argument.

We first observe that since $(\nabla u)^*$ is in $L^1_{loc}(d\sigma)$, it follows that ∇u is in $L^1(B)$ for any bounded subset of Ω, B.

Fix a point $x_0 \in \partial\Omega$ and let η be a smooth cutoff function which is one on $B_{2r}(x_0)$ and zero outside of $B_{4r}(x_0)$. We let $N(z, y)$ be the Neumann function for Ω, thus N is a fundamental solution which satisfies homogeneous Neumann boundary conditions. Dahlberg and Kenig [10] construct the Neumann function on a graph domain by reflection. At least formally, we have the representation formula,

$$\begin{aligned} \eta u(z) \;&=\; \int_{\Omega\cap B_{4r}(x_0)} N(z,y)(u(y)\Delta\eta(y) + 2\nabla u(y)\cdot\nabla\eta(y))\,dy \\ &\quad - \int_{\Delta_{4r}(x_0)} N(z,t)u(t)\frac{\partial\eta(t)}{\partial\nu}\,d\sigma(t)\,, \qquad z\in\Omega. \end{aligned} \tag{2.6}$$

We now show how to estimate each term on the right-hand side of this formula. Since $(\nabla u)^*(x)$ is in $L^1_{loc}(\partial\Omega)$, it follows that u is bounded on $\Delta_{4r}(x_0)$. As we observed above, we have $\nabla u \in L^1(\Omega\cap B_{4r}(x_0))$, and the Poincaré inequality gives that u is in $L^1(\Omega \cap B_{3r}(x_0))$. We also have that the Neumann function is locally bounded when $z \neq y$. Hence, the integrands in (2.6) are in L^1 and it is a routine matter to justify this formula. Since the map $y \to N(z, y)$ is in the Sobolev space $W^{1,2}(B_r(x_0)\cap\Omega)$, uniformly for $z \in B_{3r}(x_0) \setminus B_{2r}(x_0)$, the estimates for u and ∇u outlined above together with (2.6) imply that $\nabla u \in L^2(B_r(x_0)\cap\Omega)$. ∎

Next, we recall a classical fact from harmonic function theory, a Phragmen-Lindelöf theorem. This result is well-known and may be found in Protter and Weinberger [28, Section 9, Theorem 18]. We will need this theorem to control the behavior at infinity of solutions in our unbounded domains.

In the next result and below, we will let Ω_φ denote the sector

$$\Omega_\varphi \;=\; \{re^{i\theta} : r>0,\ |\theta-\pi/2|<\varphi\}, \quad 0<\varphi<\pi\,. \tag{2.7}$$

With this normalization, Ω_φ is a Lipschitz domain with constant $M = |\tan(\pi/2-\varphi)|$.

Theorem 2.5 (Phragmen-Lindelöf) *Let $\varphi \in (0, \pi)$. Suppose v is sub-harmonic in Ω_φ, $v = 0$ on $\partial\Omega_\varphi$ and*

$$v(z) = o(|z|^{\pi/(2\varphi)}) \quad \textit{as } |z| \to +\infty\,, \quad z\in\Omega_\varphi\,.$$

Then $v \le 0$.

We now prove uniqueness for the regularity problem for $L^p(w\,d\sigma)$, (2.5).

Lemma 2.6 *Let Ω be a Lipschitz graph domain with Lipschitz constant M. Suppose that, for $p \geq 1$, $L^p_{loc}(w\, d\sigma) \subset L^1_{loc}(d\sigma)$ and that that for all surface balls $\Delta_r(x)$ with $r > 1$, we have*

$$\left(\int_{\Delta_r(x)} w(t)^{-1/(p-1)}\, d\sigma(t)\right)^{\frac{p-1}{p}} \leq Cr^{\pi/(2\beta+\pi)}, \quad \text{if } p > 1; \tag{2.8}$$

$$\left(\inf_{t\in\Delta_r(x)} w(t)\right)^{-1} \leq Cr^{\pi/(2\beta+\pi)}, \quad \text{if } p = 1 \tag{2.9}$$

where $\beta = \arctan M > 0$. Under these conditions, if u is harmonic in Ω, $(\nabla u)^* \in L^p(w\, d\sigma)$ and $u = 0$ on $\partial\Omega$, then $u = 0$ in Ω.

Proof. Since $\beta = \arctan M$ then we have $\Omega \subset \Omega_{\tilde{\varphi}}$, with $\tilde{\varphi} = \beta + \pi/2$. Define v by

$$v(x) = \begin{cases} |u|, & \text{in } \Omega \\ 0, & \text{in } \bar{\Omega}^c. \end{cases}$$

The function v is sub-harmonic in all of $\mathbf{R}^2$ and, moreover, $v = 0$ on $\partial\Omega_{\tilde{\varphi}}$. We verify the growth condition in the Phragmen-Lindelöf Theorem 2.5. To do this, suppose that $z \in \Omega_{\tilde{\varphi}}$ and set $r = |z|$. By Lemma 2.4 we have that ∇u is in L^2 of each bounded subset of Ω, and the same is true for ∇v. Then, by combining the mean-value property for sub-harmonic functions with the Poincaré inequality, Proposition 2.2 (for the Carleson measure $d\mu(y) = \frac{1}{r}\chi_{B_r(z)}\, dy$ with respect to $d\sigma$, see (2.3)) and Hölder's inequality, in the case $p > 1$, we obtain

$$\begin{aligned} 0 \leq v(z) &\leq \frac{1}{\pi r^2}\int_{B_r(z)} v(y)\, dy \\ &\leq \frac{C}{r}\int_{B_r(z)} |\nabla v(y)|\, dy \\ &\leq C\int_{\Delta_{2r}(x)} (\nabla u)^*(t)\, d\sigma(t) \\ &\leq C\left(\int_{\Delta_{2r}(x)} (\nabla u)^*(t)^p\, w(t) d\sigma(t)\right)^{1/p}\left(\int_{\Delta_{2r}(x)} w^{-1/(p-1)}(t)\, d\sigma(t)\right)^{\frac{p-1}{p}}. \end{aligned}$$

Here, x is the projection of z onto the boundary, i.e. if $z = (x_1, \phi(x_1) + t)$, then $x = (x_1, \phi(x_1))$. Thus, under our assumption (2.8) it follows that

$$0 \leq v(z) \leq Cr^{\frac{\pi}{2\tilde{\varphi}}} \leq C|z|^{\frac{\pi}{2\tilde{\varphi}}}, \quad z \in \Omega_{\tilde{\varphi}},$$

as $0 \in \partial\Omega_{\tilde{\varphi}}$. We may now apply Phragmen-Lindelöf Theorem 2.5 and conclude that $v = 0$. The case $p = 1$ is treated in a similar fashion. ∎

Remark. Using Hölder's inequality, we see that

$$\int_{\Delta_s(x)} |f(t)|\, d\sigma(t) \leq \left(\int_{\Delta_s(x)} |f(t)|^p\, w(t)\, d\sigma(t) \right)^{1/p} \left(\int_{\Delta_s(x)} w(t)^{-1/(p-1)}\, d\sigma(t) \right)^{\frac{p-1}{p}}.$$

Thus, we have $L^p_{loc}(w\, d\sigma) \subset L^1_{loc}(d\sigma)$ provided $w^{-1/(p-1)}$ is in $L^1_{loc}(d\sigma)$. It is easy to see that this will hold for the weight $w(t) = |t|^\epsilon$ if $\epsilon < p - 1$.

Finally, we give uniqueness for the weighted Neumann problem for $L^p(w\, d\sigma)$, (2.4).

Lemma 2.7 *Let Ω be a Lipschitz graph domain with Lipschitz constant M. Suppose w satisfies the hypotheses of Lemma 2.6. If u is a solution of the Neumann problem with $(\nabla u)^* \in L^p(w\, d\sigma)$, $p \geq 1$, and $\frac{\partial u}{\partial \nu} = 0$ a.e. $\partial\Omega$, then u is constant.*

Proof. We consider v, the conjugate harmonic function to u. Since u has normal derivative zero at the boundary, by the Cauchy-Riemann equations we have that v is constant on the boundary, and we may assume this constant is 0. Now, $(\nabla v)^* \in L^p(w\, d\sigma)$ (since this is the case for u) so that v satisfies the hypotheses of Lemma 2.6 and hence it is zero. But this implies that u is constant. ∎

3 The L^2-Neumann and regularity problems with power weights

In this section we outline the proof of existence of solutions to the Neumann problem and the regularity problem for $L^2(w\, d\sigma)$ when the weight is a power of $|x|$. These results are due to Shen [30, Theorems 1.7, 1.9]. Shen's results are stated for bounded domains in three dimensions. We give a few steps of the proof to indicate the minor modifications that are needed to handle graph domains in two dimensions.

Before continuing, we record a few basic facts about the power weights $|x|^\epsilon$ and their relation to the Muckenhoupt class $A_p(d\sigma)$. A weight (that is a non-negative measurable and locally integrable function) w is a member of the class $A_p(d\sigma)$ if and only if for all surface balls $\Delta \subset \partial\Omega$, we have

$$\int_\Delta w\, d\sigma \left(\int_\Delta w^{\frac{-1}{p-1}}\, d\sigma \right)^{p-1} \leq C.$$

The best constant C in this inequality is called the A_p-constant for w. The following simple lemma tells us that $|x|^\epsilon$ is in $A_p(d\sigma)$ if and only if $-1 < \epsilon < p - 1$, see [31, p. 218].

Lemma 3.1 *For $\epsilon > -1$, the boundary measure σ_ϵ (see (2.3)) satisfies*

$$\sigma_\epsilon(\Delta_r(x)) \approx r \max(|x|, r)^\epsilon, \ \ x \in \partial\Omega.$$

The proof is omitted. Next, we establish existence of solutions in sectors, see (2.7).

Proposition 3.2 *Let ϵ satisfy $1 - \frac{\pi}{\varphi} < \epsilon < 1$, and let σ_ϵ be as in (2.3). Then, we may solve the Neumann problem (2.4) and the regularity problem (2.5) for $L^2(d\sigma_\epsilon)$ in the sector Ω_φ, and we have* 2006-06-06, changed 2ϕ to ϕ.

$$\|(\nabla u)^*\|_{L^2(\partial\Omega_\varphi, d\sigma_\epsilon)} \leq C\|f_N\|_{L^2(\partial\Omega_\varphi, d\sigma_\epsilon)} \qquad \textit{(for (2.4))},$$

$$\|(\nabla u)^*\|_{L^2(\partial\Omega_\varphi, d\sigma_\epsilon)} \leq C\|\tfrac{df_D}{d\sigma}\|_{L^2(\partial\Omega_\varphi, d\sigma_\epsilon)} \qquad \textit{(for (2.5))}.$$

Proof. The proof goes along the same lines as the proof of Proposition 4.3, which deals with the mixed problem in sectors, with the role of Theorem 4.2 being played by the classical results on the regularity and Neumann problems for the upper half plane. We omit the argument here in order to save space. ∎

2006-06-06, new proposition

We will need the following estimates from Shen [30].

Proposition 3.3 *Let Ω be a standard Lipschitz graph domain and suppose that $(\nabla u)^*$ is in $L^2(d\sigma_\epsilon)$. There is a constant $C = C(\|\phi'\|_\infty)$ so that*

$$\begin{aligned}
\int_{\partial\Omega} (\nabla u)^*(x)^2 \, d\sigma_\epsilon(x) &\leq C \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma_\epsilon \\
\int_{\partial\Omega} (\nabla u)^*(x)^2 \, d\sigma_\epsilon(x) &\leq C \int_{\partial\Omega} \left|\frac{du}{ds}\right|^2 d\sigma_\epsilon
\end{aligned}$$

Proof. The arguments of Shen [30] carry over with minor changes to graph domains in two dimensions. The estimates for the Neumann function in Shen's work (see (3.2) on page 2850 of [30]) may be proven in graph domains using the ideas of Dahlberg and Kenig [10]. Also, see the argument below where we establish the estimate (6.6) for the Green's function for the mixed problem. ∎

Remark. The estimate for the regularity problem holds for a larger class of weights than power weights. The estimate for the Neumann problem holds only for power weights. This estimate also holds for power weights with a small negative exponent. This may be proven using the Rellich identity. Shen's argument may also be used provided one can show that the Neumann function is Hölder with exponent δ for some $\delta > 1/2$.

The main result of this section is the following theorem.

Theorem 3.4 (Z. Shen) *Let Ω be a standard Lipschitz graph domain with $\|\phi'\|_\infty < \infty$. For ϵ satisfying $0 \le \epsilon < 1$ and for σ_ϵ as in (2.3), we have that the Neumann problem (2.4) and the regularity problem (2.5) for $L^2(d\sigma_\epsilon)$ are uniquely solvable.*

Proof. We apply the method of layer potentials as in Verchota [35]. Note that for $\epsilon \in (-1, 1)$, $|x|^\epsilon$ is an $A_2(d\sigma)$ weight so that singular integrals are bounded on $L^2(\partial\Omega, d\sigma_\epsilon)$. The estimates of Proposition 3.3 imply that

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma_\epsilon \approx \int_{\partial\Omega} \left|\frac{du}{d\sigma}\right|^2 d\sigma_\epsilon\,, \tag{3.1}$$

if u is harmonic in Ω with $(\nabla u)^* \in L^2(d\sigma_\epsilon)$ and $0 \le \epsilon < 1$. By Proposition 3.2, we can solve the Neumann and regularity problems for $L^2(\partial\Omega_\varphi, d\sigma_\epsilon)$ for $0 \le \epsilon < 1$. The estimate (3.1) and the method of continuity (see Brown [4, Lemma 1.16] and Gilbarg and Trudinger [15, Theorem 5.2]) now lead to the existence of a solution in general graph domains.

Uniqueness follows from Lemma 2.6 or 2.7. It is easy to check that the measure $d\sigma_\epsilon$ satisfies the growth condition of Lemma 2.6. ∎

4 The mixed problem with power weights.

The main result of this section is Theorem 4.6 where we prove existence and uniqueness for the solutions of the weighted mixed problem in L^2 on a Lipschitz graph domain. There will be a restriction on the size of the Lipschitz constant. We first discuss the unweighted case. The key estimate is obtained using the Rellich identity with a vector field α so that $\alpha \cdot \nu$ changes sign as we move from D to N. This technique is applied to the Mixed problem in [4]. A similar technique is applied to a question about the wave equation by Lagnese [22]. This earlier work requires the domain to be smooth and thus the sets N and D must be consist of disjoint connected components of the boundary.

Lemma 4.1 (Rellich Estimates for the unweighted mixed problem) *Let $\Omega = \{x_2 > \phi(x_1)\}$, N, D be a standard Lipschitz graph domain for the mixed problem, see (1.2). Suppose that $\phi_{x_1} > \delta > 0$ on N and $\phi_{x_1} < -\delta < 0$ on D.*

Then, if u is harmonic in Ω and $(\nabla u)^* \in L^2(\partial\Omega, d\sigma)$, we have

$$\int_D \left(\frac{du}{d\sigma}\right)^2 d\sigma + \int_N \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma \approx \int_N \left(\frac{du}{d\sigma}\right)^2 d\sigma + \int_D \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma. \tag{4.1}$$

Proof. The proof follows from the Rellich identity with vector field $\alpha = e_1$, see Brown [4, Lemma 1.7]. ∎

Theorem 4.2 *Let $\delta > 0$ and Ω, N, D be a standard Lipschitz graph domain for the mixed problem, see (1.2). Suppose that $\phi_{x_1} > \delta > 0$ on N and $\phi_{x_1} < -\delta < 0$ on D. Then, if $g_N \in L^2(N, d\sigma)$ and $dg_D/d\sigma \in L^2(D, d\sigma)$, there exists a unique solution of the mixed problem (1.1) for $L^2(d\sigma)$ in Ω. Moreover, the solution v satisfies*

$$\|(\nabla v)^*\|_{L^2(d\sigma)} \leq C \left(\|\frac{dg_D}{d\sigma}\|_{L^2(D,d\sigma)} + \|g_N\|_{L^2(N,d\sigma)} \right). \tag{4.2}$$

Remark. In particular, we have that the unweighted Mixed problem is uniquely solvable (with estimates for $(\nabla v)^*$) in all convex sectors Ω_ψ, $0 < \psi < \pi/2$, see (2.7).

Proof. This is an extension of the results of Brown [4] to two dimensions and to Lipschitz graph domains. The first step is to observe that the equivalence (4.1) quickly leads to uniqueness. For if u is a solution of (1.1) with $u = 0$ on D and $\frac{\partial u}{\partial \nu} = 0$ on N, then (4.1) implies $u = 0$ on $\partial\Omega$. Lemma 2.6 now yields $u = 0$ in Ω.

To prove existence, we find solutions in a sector. This is easy by symmetry and can be done as in Brown [4, Lemma 1.16]. Then we may use the method of continuity and again (4.1) to establish solutions in more general Lipschitz graph domains. ∎

We now turn our attention to the weighted mixed problem in sectors that are not necessarily convex. For the boundary of a sector Ω_φ as in (2.7), we write: $\partial\Omega_\varphi = D_\varphi \cup N_\varphi$, with D and N as in (1.2).

Proposition 4.3 *Let $\varphi \in (\pi/2, \pi)$ and suppose that $1 - \frac{\pi}{2\varphi} < \epsilon < 1$. Let σ_ϵ be as in (2.3). Then, if $f_N \in L^2(N_\varphi, d\sigma_\epsilon)$ and $df_D/d\sigma \in L^2(D_\varphi, d\sigma_\epsilon)$, there exists a unique solution u of the mixed problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_\varphi \\ u = f_D & \text{on } D_\varphi \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N_\varphi \\ (\nabla u)^* \in L^2(\partial\Omega_\varphi, d\sigma_\epsilon). \end{cases} \tag{4.3}$$

Moreover, u satisfies

$$\int_{\partial\Omega_\varphi} (\nabla u)^*(x)^2 \, d\sigma_\epsilon \leq C \left(\int_{N_\varphi} |f_N(x)|^2 \, d\sigma_\epsilon + \int_{D_\varphi} \left| \frac{df_D}{d\sigma}(x) \right|^2 \, d\sigma_\epsilon \right) \tag{4.4}$$

Proof. We use a conformal map to reduce the weighted Mixed problem (4.3) to an unweighted Mixed problem on a convex sector, then we apply Theorem 4.2.

Let $s = 1 - \epsilon$ with ϵ as in the hypothesis, so that we have

$$0 < s < \pi/2\varphi < 1. \tag{4.5}$$

Define $\phi_s(z)$ to be the conformal map

$$\eta = \phi_s(z) = i(-iz)^s.$$

Thus, ϕ_s maps Ω_φ to $\Omega_{s\varphi}$, D_φ to $D_{s\varphi}$ and N_φ to $N_{s\varphi}$. Note that if we let $\partial\phi_s$ denote the complex derivative of ϕ_s, that is

$$\partial\phi_s(z) = \frac{1}{2}\left(\frac{\partial\phi_s}{\partial x_1} + \frac{1}{i}\frac{\partial\phi_s}{\partial x_2}\right) \tag{4.6}$$

we have

$$\frac{1}{|\partial\phi_s|}d\sigma = \frac{1}{s}d\sigma_\epsilon\,. \tag{4.7}$$

On account of (4.5) it follows that Theorem 4.2 applies to $\Omega_{s\varphi}$.

Given f_D and f_N as in (4.3), we define g_D and g_N on $\partial\Omega_{s\varphi}$ as follows:

$$g_N(\eta) = \frac{f_N}{|\partial\phi_s|} \circ \phi_s^{-1}(\eta)\,; \quad g_D(\eta) = f_D \circ \phi_s^{-1}(\eta).$$

We verify that g_N and g_D satisfy the hypotheses of Theorem 4.2. Indeed, it is immediate to see that

$$\int_{N_{s\varphi}} |g_N|^2 d\sigma = \frac{1}{s}\int_{N_\varphi} |f_N|^2 d\sigma_\epsilon\,. \tag{4.8}$$

Similarly, we have

$$\int_{N_{s\varphi}} \left|\frac{dg_D}{d\sigma}\right|^2 d\sigma = \frac{1}{s}\int_{N_\varphi} \left|\frac{df_D}{d\sigma}\right|^2 d\sigma_\epsilon\,. \tag{4.9}$$

By Theorem 4.2 the unweighted Mixed problem on $\Omega_{s\varphi}$ has a unique solution v which satisfies (4.2). We now pull this solution back to Ω_φ by defining $u = v \circ \phi_s$. Then, u is harmonic in Ω_φ and satisfies:

$$u = f_D \quad \text{on } D_\varphi\,; \qquad \frac{\partial u}{\partial \nu} = f_N \quad \text{on } N_\varphi\,.$$

Moreover, on account of (4.2), (4.8) and (4.9) we have

$$\|(\nabla v)^*\|_{L^2(\partial\Omega_{s\varphi}, d\sigma)} \le C\int_{N_\varphi} |f_N(x)|^2\, d\sigma_\epsilon + \int_{D_\varphi}\left|\frac{df_D}{ds}(x)\right|^2 d\sigma_\epsilon. \tag{4.10}$$

Now we consider non-tangential maximal function estimates for ∇u. We apply the Cauchy integral formula to the complex derivative of u (which is analytic in Ω_φ) and obtain:

$$\partial u(z) \;=\; \frac{1}{2\pi i}\int_{\partial\Omega_\varphi} \frac{\partial u(\zeta)}{z-\zeta}\, d\zeta \;=\; \frac{1}{2\pi i}\int_{\partial\Omega_\varphi} \frac{\partial\phi_s(\zeta)\partial v(\phi_s(\zeta))}{z-\zeta}\, d\zeta\,, \qquad z \in \Omega_\varphi\,.$$

It follows

$$(\nabla u)^*(x) = 2(\partial u)^*(x) \le \Big(K_\varphi(\partial\phi_s \cdot (\partial v)\circ\phi_s)\Big)^*(x)\,, \quad \text{a.e. } x \in \partial\Omega_\varphi\,,$$

where K_φ denotes the Cauchy integral on Ω_φ. By the theorem of Coifman, McIntosh and Meyer [7] on the boundedness of the Cauchy integral we have

$$\|(K_\varphi \psi)^*\|_{L^2(\partial\Omega_\varphi, d\sigma_\epsilon)} \leq C\|\psi\|_{L^2(\partial\Omega_\varphi, d\sigma_\epsilon)}.$$

This uses that $d\sigma_\epsilon$ is in $A_2(d\sigma)$. Combining these last two inequalities we obtain

$$\|(\nabla u)^*\|_{L^2(\partial\Omega_\varphi, d\sigma_\epsilon)} \leq C\|\partial\phi_s \cdot (\partial v)\circ\phi_s)\|_{L^2(\partial\Omega_\varphi, d\sigma_\epsilon)} = C\|\partial v\|_{L^2(\partial\Omega_{s\varphi}, d\sigma)}, \tag{4.11}$$

where the last equality was obtained by performing the change of variables $\phi_s(\zeta) = \eta$, see also (4.7). This, together with (4.10) yields (4.4).

The argument is reversible, so we may conclude uniqueness in Ω_φ from uniqueness in $\Omega_{s\varphi}$. ∎

Remark. It is well-known that non-tangential maximal function estimates for harmonic functions behave nicely under conformal mapping, see Kenig [19] and Jerison and Kenig [17]. The previous Lemma, however, considers the non-tangential maximal function of the gradient. The estimates in this case appear to be more involved.

We now construct holomorphic vector fields which allow us to use the Rellich identity of Lemma 2.3 to obtain Rellich estimates for the weighted mixed problem.

Lemma 4.4 *Suppose $\Omega = \{x_2 > \phi(x_1)\}$, N, D is a standard Lipschitz graph domain for the mixed problem, with Lipschitz constant M. Let $\beta = \arctan M > 0$. Assume $\beta < \pi/4$.*

Then, for $2\beta/(\pi - 2\beta) < \epsilon < 1$ there exist $\beta_0 = \beta_0(\epsilon, M)$, $\beta < \beta_0 < (\pi - 2\beta)/2$, and a complex number $a = e^{i\lambda}$ such that the vector field $\alpha(z) = (Re(az^\epsilon), Im(az^\epsilon))$ satisfies

$$-|x|^\epsilon \leq \alpha(x)\cdot\nu(x) < -|x|^\epsilon \sin(\beta_0 - \beta), \qquad x \in N; \tag{4.12}$$

$$|x|^\epsilon \geq \alpha(x)\cdot\nu(x) > |x|^\epsilon \sin(\beta_0 - \beta), \qquad x \in D. \tag{4.13}$$

Proof. The outer unit normal ν lies in $\{(\cos\varphi, \sin\varphi) : -\pi/2 - \beta \leq \varphi \leq -\pi/2 + \beta\}$. On account of (1.2) and (1.3) we have that N is contained in the sector $\{x = re^{i\theta} : -\beta < \theta < \beta\}$, whereas D is contained in $\{x = re^{i\theta} : \pi - \beta < \theta < \pi + \beta\}$. Thus, in order to have $\alpha\cdot\nu < 0$ on N we need $\alpha/|\alpha| = (\cos\psi, \sin\psi)$ for some $\psi \in (-2\pi + \beta, -\pi - \beta)$, whereas $\alpha\cdot\nu > 0$ on D requires $\psi \in (-\pi + \beta, -\beta)$. To obtain a strictly negative upper bound for $\alpha\cdot\nu$ on N and a strictly positive lower bound for $\alpha\cdot\nu$ on D, we pick β_0 (to be selected later) so that $\beta < \beta_0 < \pi/2$ and then, with the same notations as above, require that, for $x = re^{i\theta} \in N$, $\psi(\theta)$ lie in $[\beta_0, \pi - \beta_0]$ whereas, for $x \in D$, we require that $\psi(\theta)$ lie in $[\pi + \beta_0, 2\pi - \beta_0]$. To this end, given ϵ as in the hypothesis, we let let $\psi(\theta)$ be a linear function with slope ϵ and choose β_0 so that ψ maps the point β to $\pi - \beta_0$ and the point $\pi - \beta$ to $\pi + \beta_0$. Writing $\psi(\theta) = \epsilon\theta + \lambda$, we define $\alpha(z) = e^{i\lambda}z^\epsilon$, that is

$$\alpha(re^{i\theta}) = r^\epsilon e^{i\psi(\theta)}$$

where $\lambda = \pi - \pi\beta_0/(\pi - 2\beta)$ and $\epsilon = 2\beta_0/(\pi - 2\beta)$ (note that the latter defines β_0). This construction, however, grants that $\alpha \cdot \nu$ has the desired sign only near the endpoints $\theta = \beta$ (for N) and $\theta = \pi - \beta$ (for D). In order to make sure that $\alpha \cdot \nu$ keeps the desired sign all the way through the two other endpoints we need to restrict the range for the selection of β_0 to:

$$\beta < \beta_0 < \frac{\pi}{2} - \beta\,.$$

Then the angle between $\alpha(x)$ and $\nu(x)$ will lie in the intervals $(-3\pi/2 + (\beta_0 - \beta), -\pi/2 - (\beta_0 - \beta))$, for $x \in N$, and in $(-\pi/2 + (\beta_0 - \beta), \pi/2 - (\beta_0 - \beta))$, for $x \in D$, so that (4.12) and (4.13) hold. ∎

Proposition 4.5 (Weighted Rellich estimates for the mixed problem) *Let Ω, N, D be a standard Lipschitz graph domain for the mixed problem (1.1), with Lipschitz constant $M < 1$. Let $\beta = \arctan M$. Then, for σ_ϵ as in (2.3), for ϵ in the range $2\beta/(\pi - 2\beta) < \epsilon < 1$ and for u harmonic with $(\nabla u)^* \in L^2(d\sigma_\epsilon)$, we have*

$$\int_{\partial\Omega} (\nabla u)^*(x)^2\, d\sigma_\epsilon \le C\left[\int_N \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma_\epsilon + \int_D \left(\frac{du}{d\sigma}\right)^2 d\sigma_\epsilon\right].$$

Proof. We use the identity of Lemma 2.3 with the vector field constructed in Lemma 4.4 and standard manipulations involving the boundary terms as in Brown [4, Lemma 1.7] too obtain the estimate

$$\int_{\partial\Omega} |\nabla u|(x)^2\, d\sigma_\epsilon \le C\left[\int_N \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma_\epsilon + \int_D \left(\frac{du}{d\sigma}\right)^2 d\sigma_\epsilon\right].$$

The key point is that since $\alpha \cdot \nu$ changes sign as we pass from N to D, we can estimate the full gradient of u on the boundary by the data for the mixed problem.

To obtain the estimate for the non-tangential maximal function, we argue as in Proposition 4.3. We may represent the holomorphic function ∂u using the Cauchy kernel

$$\partial u(x) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\partial u(y)}{x - y} dy. \tag{4.14}$$

Since σ_ϵ is an A_2 weight with respect to σ for $-1 < \epsilon < 1$, the theorem of Coifman, McIntosh and Meyer [7] implies the non-tangential maximal estimate $\|(\partial u)^*\|_{L^2(\sigma_\epsilon)} \le C\|\partial u\|_{L^2(\sigma_\epsilon)}$. The estimate of the proposition follows from this and the corresponding estimate for $\bar\partial u$. ∎

Theorem 4.6 *Suppose Ω, N and D is a standard Lipschitz graph domain for the mixed problem, with Lipschitz constant $M < 1$. Let $\beta = \arctan(M)$. Given ϵ which satisfies*

$$2\beta/(\pi - 2\beta) < \epsilon < 1,$$

there is a unique solution u to the $L^2(d\sigma_\epsilon)$ mixed problem (1.1). Moreover, u satisfies

$$\int_{\partial\Omega} (\nabla u)^*(x)^2 \, d\sigma_\epsilon \leq \left(\int_D \left| \frac{df_D}{d\sigma} \right|^2 d\sigma_\epsilon + \int_N |f_N|^2 \, d\sigma_\epsilon \right).$$

Proof. Using the estimate of Proposition 4.5 and the existence result for sectors in Proposition 4.3, the method of continuity (Brown [4], Gilbarg and Trudinger [15]), implies the existence of a solution to the mixed problem with data in $L^2(d\sigma_\epsilon)$.

Next, we consider uniqueness. If u is a solution of the mixed problem with zero data, then the Rellich estimates or Proposition 4.5 imply that u is a solution of the regularity and Neumann problems for $L^2(d\sigma_\epsilon)$ with zero data. By the results in Section 2 it follows that u is zero. ∎

5 The regularity and Neumann problems in $H^1(d\sigma_\epsilon)$.

In this section we assert the existence of solutions for the Neumann problem when the data is in $H^1(d\sigma_\epsilon)$, and for the regularity problem when the data has one derivative in $H^1(d\sigma_\epsilon)$. The proof of these results follows the work of Dahlberg and Kenig [10]; thus, we shall be brief. We first recall the definition of the Hardy spaces $H^1(d\sigma_\epsilon)$ and $H^{1,1}(d\sigma_\epsilon)$.

Let $\epsilon > -1$. We say that a is an $H^1(d\sigma_\epsilon)$-*atom for* $\partial\Omega$ if a is supported on a surface ball $\Delta_s(x)$, $\int a \, d\sigma = 0$ and $\|a\|_\infty \leq [\sigma_\epsilon(\Delta_s(x))]^{-1}$. We remark that for $\epsilon \leq 0$, $L^1(d\sigma_\epsilon) \subset L^1_{loc}(d\sigma)$. Thus we define the space $H^1(d\sigma_\epsilon)$, for $\epsilon \leq 0$, to be the collection of functions that are represented as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x) \tag{5.1}$$

where $\{a_j\}_{j\in\mathbf{N}}$ is a sequence of $H^1(d\sigma_\epsilon)$-atoms for $\partial\Omega$ and the coefficients $\{\lambda_j\}$ satisfy $\sum_{j=1}^\infty |\lambda_j| < \infty$. The sum for f in (5.1) converges in $L^1(d\sigma_\epsilon)$. the sum converges in $L^1(d\sigma_\epsilon)$. The $H^1(d\sigma_\epsilon)$-norm is defined by

$$\|f\|_{H^1(d\sigma_\epsilon)} = \inf\{\sum |\lambda_j|\} \tag{5.2}$$

where the infimum is taken over all possible representations of f. Note that while $H^1(d\sigma_\epsilon)$-atoms are defined for $\epsilon > -1$, we consider (and need for our application in Theorem 7.2) the space $H^1(d\sigma_\epsilon)$ only for $\epsilon \leq 0$. This allows us to avoid having to define spaces of distributions on Lipschitz graph domains. (See Coifman and Weiss [8] or Stromberg and Torchinsky [32] for a discussion of these spaces.)

Let $\epsilon \leq 0$. We say that A is an $H^{1,1}(d\sigma_\epsilon)$-*atom for* $\partial\Omega$ if for some x_0 in $\partial\Omega$

$$A(x) = \int_{x_0}^{x} a(t) d\sigma(t), \qquad x \in \partial\Omega$$

where a is an $H^1(d\sigma_\epsilon)$-atom, and $\int_{x_0}^x$ denotes integration along the portion of the boundary $\partial\Omega$ with endpoints x_0 and x. Given an $H^1(d\sigma_\epsilon)$-atom a, the integral above defines A uniquely up to an additive constant. For $\epsilon \le 0$, we define the space $H^{1,1}(d\sigma_\epsilon)$ to be sums of the form

$$F(x) = \sum_{j=1}^{\infty} \lambda_j A_j(x)$$

where the coefficients satisfy $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. The norm of F in $H^{1,1}(\sigma_{\epsilon'})$, $\|F\|_{H^{1,1}(d\sigma_\epsilon)}$, is defined to be the infimum of $\sum_{j=1}^{\infty} |\lambda_j|$ over all possible representations of F as sums of atoms. If we choose the base point x_0 for each atom to be zero, then we have that the sum defining F converges in the norm given by

$$\sup |x|^{-\epsilon} |F(x)|.$$

Furthermore, it is easy to see that we have

$$\|F\|_{H^{1,1}(d\sigma_\epsilon)} = \|\frac{dF}{d\sigma}\|_{H^1(d\sigma_\epsilon)}.$$

By the *Neumann problem for* $H^1(d\sigma_\epsilon)$ we mean the Neumann problem for $L^1(d\sigma_\epsilon)$, see (2.4), where the data f_N is now taken from $H^1(d\sigma_\epsilon)$. By the *regularity problem for* $H^{1,1}(d\sigma_\epsilon)$ we mean the regularity problem for $L^1(d\sigma_\epsilon)$, see (2.5), where the data f_D is taken from $H^{1,1}(d\sigma_\epsilon)$.

For future reference, we also define spaces on subsets of the boundary. A function f is in $H^1(N, d\sigma_\epsilon)$ if and only if f is the restriction to N of a function in $H^1(d\sigma_\epsilon)$. Such functions can be written as sums of atoms that are restrictions to N of $H^1(d\sigma_\epsilon)$-atoms for $\partial\Omega$. The space $H^{1,1}(D, d\sigma_\epsilon)$ is defined in a similar fashion. See Chang, Krantz and Stein [6], Sykes [33], and Sykes and Brown [34] for additional works that study Hardy spaces on domains.

The main result of this section is the following theorem. We omit the proof since it is quite similar to the argument for the mixed problem in the following section (Theorem 7.2).

Theorem 5.1 *Let Ω be a standard Lipschitz graph domain with Lipschitz constant M and let $\epsilon_0(M) < \epsilon' \le 0$ be small. Then the $H^1(d\sigma_{\epsilon'})$-Neumann and the $H^{1,1}(d\sigma_{\epsilon'})$-regularity problems are uniquely solvable. The solutions satisfy, respectively,*

$$\begin{aligned}
\|u\|_{H^{1,1}(d\sigma_{\epsilon'})} + \int_{\partial\Omega} (\nabla u)^*(x)\, d\sigma_{\epsilon'} &\le C \left\| \frac{\partial u}{\partial \nu} \right\|_{H^1(d\sigma_{\epsilon'})} \\
\left\| \frac{\partial u}{\partial \nu} \right\|_{H^1(d\sigma_{\epsilon'})} + \int_{\partial\Omega} (\nabla u)^*(x)\, d\sigma_{\epsilon'} &\le C \|u\|_{H^{1,1}(d\sigma_{\epsilon'})}.
\end{aligned}$$

Remark. We do not assert uniqueness when $\epsilon' > 0$. In the rest of this paper, we will only use the existence of a solution for the $H^1(d\sigma_{\epsilon'})$-Neumann problem in the case when $\epsilon' \leq 0$. (In order to fully treat the case $\epsilon' > 0$, one needs to give a different definition of the normal derivative at the boundary. For $\epsilon' > 0$ the $H^1(d\sigma_{\epsilon'})$-boundary data may not be in $L^1_{loc}(d\sigma)$ and hence fail to be a function. See Brown [5] and Fabes, Mendez and Mitrea [13] for a treatment of the Neumann problem with data which is not locally integrable).

6 The mixed problem with atomic data.

By the *mixed problem for* $H^1(d\sigma_{\epsilon'})$ we mean the mixed problem for $L^1(d\sigma_{\epsilon'})$, see (1.1), where the Dirichlet data f_D is taken from $H^{1,1}(D, d\sigma_{\epsilon'})$ and the Neumann data f_N is in $H^1(N, d\sigma_{\epsilon'})$. In this section, we consider the mixed problem where the Neumann data is an $H^1(N, d\sigma_{\epsilon'})$ atom and the Dirichlet data is zero. Since atoms lie in $L^2(d\sigma_\epsilon)$ for $\epsilon > -1$, by Theorem 4.6 we have existence and uniqueness of the solution to the mixed problem for $L^2(d\sigma_\epsilon)$ with these data. Our first goal is to show that the gradient of this solution also has non-tangential maximal function in $L^1(d\sigma_{\epsilon'})$ for ϵ' near zero.

Theorem 6.1 *Suppose Ω, N, D is a standard Lipschitz graph domain for the mixed problem with Lipschitz constant $M < 1$ and set $\beta = \tan^{-1} M$. Then, there is $\delta = \delta(M)$ with $1 > \delta > 0$ so that, for ϵ' satisfying $\frac{4\beta-\pi}{2(\pi-2\beta)} < \epsilon' < \delta$ we may solve the mixed problem (1.1) for $H^1(d\sigma_{\epsilon'})$ with zero Dirichlet data and with Neumann data an $H^1(N, d\sigma_{\epsilon'})$-atom, a. The solution u satisfies the estimate*

$$\int_{\partial\Omega} (\nabla u)^*(x)\, d\sigma_{\epsilon'} \leq C(M, \epsilon')\,. \tag{6.1}$$

In addition, we have

$$\|u\|_{H^{1,1}(d\sigma_{\epsilon'})} + \left\|\frac{\partial u}{\partial \nu}\right\|_{H^1(d\sigma_{\epsilon'})} \leq C(M, \epsilon'). \tag{6.2}$$

As a step towards the proof of Theorem 6.1, we construct a Green's function for the mixed problem using the method of reflections–an old idea that was used by Dahlberg and Kenig [10] to obtain a similar result for the Neumann and regularity problems. The estimates for the Green's function are a consequence of the Hölder regularity of weak solutions of divergence-form equations with bounded measurable coefficients.

We begin by constructing a bi-Lipschitz map, $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ with $\Phi(\Omega) = Q$ where Q is the first quadrant, $Q = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$. We also require that $\Phi(N) = \{(x_1, x_2) : x_1 \geq 0, x_2 = 0\}$ and $\Phi(D) = \{(x_1, x_2) : x_1 = 0, x_2 > 0\}$. On Q we now define the operator $L = \mathrm{div} A\nabla$ where the coefficient matrix A has bounded entries (these are first-order derivatives of Φ), so that u satisfies

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = f_N, & \text{on } N \\ u = f_D, & \text{on } D \end{cases}$$

if and only if the function v defined by $v = u \circ \Phi^{-1}$ satisfies

$$\begin{cases} Lv = 0, & \text{in } Q \\ A\nabla v \cdot \nu = f_N \circ \Phi^{-1} & \text{on } \Phi(N) \\ v = f_D \circ \Phi^{-1} & \text{on } \Phi(D) \end{cases}$$

Now, we extend the coefficients of L by reflection, so that $Lv = 0$ if and only if $L(v \circ R_j) = 0$ where R_1 and R_2 are the reflections $R_1(x_1, x_2) = (-x_1, x_2)$ and $R_2(x_1, x_2) = (x_1, -x_2)$. Next, letting G denote the Green's function for L in Q, we set

$$\mathcal{N}(z, w) = G(z, w) - G(z, R_1 w) + G(z, R_2 w) - G(z, R_1 R_2 w)\,, \quad z, w \in Q$$

and observe that $\mathcal{N}$ is a Green's function for the mixed problem for L in Q. Recall that we may find a Green's function for L in Q which satisfies

$$|G(z, w)| \le C(1 + |\log|z - w||)\,, \quad z, w \in Q,$$

see Kenig and Ni [21]. We then observe that if $|z - \zeta| = 1$ and $|\zeta - w| < 1/2$, then $|G(z, \zeta) - G(z, w)| \le C|\zeta - w|^\delta$, where $0 < \delta < 1$, $\delta = \delta(M)$. This is a standard estimate of Hölder continuity for solutions of divergence-form elliptic operators. Finally, by rescaling, we obtain

$$|G(z, \zeta) - G(z, w)| \le C\left(\frac{|\zeta - w|}{|z - \zeta|}\right)^\delta, \qquad \text{if } |\zeta - w| < \frac{1}{2}|z - \zeta|. \tag{6.3}$$

The latter immediately implies the same estimate for $\mathcal{N}$ in Q, namely

$$|\mathcal{N}(z, \zeta) - \mathcal{N}(z, w)| \le C\left(\frac{|\zeta - w|}{|z - \zeta|}\right)^\delta \quad \text{if } z, \zeta, w \in \overline{Q} \text{ and } |\zeta - w| < \frac{1}{2}|z - \zeta| \tag{6.4}$$

We will need an additional estimate for $\mathcal{N}(z, \zeta)$ when $\zeta \in Q$ is near $\Phi(D)$. Since we have $\mathcal{N}(z, x) = 0$ if $z \in Q$ and $x \in \Phi(D)$, it follows by continuity that $\mathcal{N}$ is small near $\Phi(D)$. More precisely, let $\zeta \in Q$ and suppose that $\hat{x}$ is a point on $\Phi(D)$ for which $|\hat{x} - \zeta| = \operatorname{dist}(\zeta, \Phi(D))$. For $z \in Q$, we have $\mathcal{N}(z, \hat{x}) = 0$ and (6.4) implies

2006-06-14, Green's function in Ω and Q now have different notations, since they are different functions.

$$|\mathcal{N}(z, \zeta)| \le C\frac{|\zeta|^\delta}{|z - \zeta|^\delta}\,, \quad \text{if } z, \zeta \in Q \text{ and } \operatorname{dist}(\zeta, D) < \frac{1}{2}|z - \zeta|, \tag{6.5}$$

(here we have used the fact that $0 \in \partial Q$). Finally, we define the Green's function for the mixed problem by

$$\mathcal{M}(x, y) = \mathcal{N}(\Phi(x), \Phi(y)).$$

Since Φ is bi-Lipschitz the estimates (6.4) and (6.5) imply similar results for $\mathcal{M}$ in Ω,

$$|\mathcal{M}(x,\zeta) - M(z,w)| \leq C\left(\frac{|\zeta - w|}{|z-\zeta|}\right)^{\delta},$$

$$\text{if } z,\zeta,w \in \bar{\Omega} \text{ with } |\zeta - w| < \frac{1}{2}|z-\zeta| \quad (6.6)$$

$$|\mathcal{M}(z,\zeta)| \leq C\frac{|\zeta|^{\delta}}{|z-\zeta|^{\delta}}$$

$$\text{if} z,\zeta \in \Omega \text{ and } \operatorname{dist}(\zeta, D) < \frac{1}{2}|z-\zeta|. \quad (6.7)$$

These estimates are a key ingredient in the study of the behavior of the solution of the mixed problem with atomic Neumann data.

Lemma 6.2 *Assume $\epsilon' > -1$. Let u be a solution of the mixed problem (1.1) for $L^1(d\sigma_{\epsilon'})$, where the Neumann data is an $H^1(N, d\sigma_{\epsilon'})$-atom, a, and the Dirichlet data is zero. Then, for any integer $k \geq 1$, u satisfies*

$$|u(z)| \leq C\frac{\rho^{\delta}}{|z-x_a|^{\delta}}\frac{\sigma(\Delta_\rho(x_a))}{\sigma_{\epsilon'}(\Delta_\rho(x_a))}, \quad z \in \Omega\,, \quad |z - x_a| \geq 2^k\rho.$$

Here, $0 < \delta < 1$ is as in the estimate for the Green's function for the mixed problem (6.3) and $\Delta_\rho(x_a)$ is the surface ball where a is supported.

Proof. We consider two cases: 1) $\Delta_\rho(x_a) \subset N$ and 2) $\Delta_\rho(x_a) \cap D \neq \emptyset$.

In case 1), we have

$$u(z) = \int_N \mathcal{M}(z,x)a(x)\,d\sigma(x) = \int_N (\mathcal{M}(z,x) - \mathcal{M}(z,x_a))a(x)\,d\sigma(x)\,, \quad z \in \Omega$$

where the second identity uses the fact that the atom a has mean value zero. Next, we use the continuity of $\mathcal{M}$ from (6.6) to obtain

$$|u(z)| \leq C\frac{\rho^{\delta}}{|z-x_a|^{\delta}}\int_N |a|\,d\sigma, \quad \text{for } z \in \Omega\,,\ |z-x_a| > 2^k\rho. \quad (6.8)$$

Finally, the normalization of a in the definition of an atom implies that

$$\int_N |a|\,d\sigma \leq \frac{\sigma(\Delta_\rho(x_a))}{\sigma_{\epsilon'}(\Delta_\rho(x_a))}\,.$$

This completes the proof in case 1.

In case 2), we do not have $\int_N a\,d\sigma = 0$. However, estimate (6.7) yields

$$|u(z)| \leq C\frac{\rho^{\delta}}{|z-x_a|^{\delta}}\int_N |a|\,d\sigma\,,$$

and then we use the normalization of a to conclude the proof. ∎

Before proceeding, we need a few technical results. In this Lemma and below, given a point $x_a \in \partial\Omega$, we consider a ball $B_\rho(x_a)$ and a boundary ball $\Delta_\rho(x_a)$, and set: $R_k = B_{2^{k+1}\rho}(x_a) \setminus B_{2^k\rho}(x_a)$, $\tilde{R}_k = B_{2^{k+2}\rho}(x_a) \setminus B_{2^{k-1}\rho}(x_a)$, $\Delta_k = \Delta_{2^k\rho}(x_a)$ and $\Sigma_k = \Delta_{k+1} \setminus \Delta_k$. With these definitions, we can now state

Lemma 6.3 *Let $\lambda \in \mathbf{R}$ and set $\alpha(z) = (\mathrm{Re}(e^{i\lambda}(z)^\epsilon), \mathrm{Im}(e^{i\lambda}(z)^\epsilon))$. Then, we have*

$$\int_{R_k} |\alpha|^p \, dy \leq C 2^k \rho \int_{\Sigma_k} |\alpha|^p \, d\sigma \leq C 2^k \rho \sigma_{\epsilon p}(\Sigma_k)$$

provided $\epsilon p > -1$.

Proof. The proof is a computation and we omit the details. ∎

We now take a brief detour to discuss solutions of the mixed problem in the energy sense. Let B be a ball with center in Ω. We say that u is an *energy solution of the mixed problem in B*,

$$\begin{cases} \Delta u = 0 & \text{in } B \cap \Omega \\ u = 0 & \text{on } D \cap B \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } N \cap B \end{cases}$$

if u lies in the Sobolev space $W^{1,2}(B \cap \Omega)$, u vanishes on $D \cap B$ and for every v that lies in $W^{1,2}(B \cap \Omega)$ and vanishes on $\partial(B \cap \Omega) \setminus N$ we have

$$\int_{B\cap\Omega} \nabla u \cdot \nabla v \, dy = 0.$$

Using a Carleson measure argument, see Proposition 2.2, it is not difficult to see that if u is a solution of the mixed problem in $L^2(d\sigma_\epsilon)$ then $|\nabla u|^2$ is integrable on bounded subsets of Ω, provided $\epsilon < 1$. This shows that a solution for the mixed problem in $L^2(d\sigma_\epsilon)$ is, in particular, an energy solution.

In the next lemma, we use $⨍_E f \, dx$ to denote the average, $⨍_E f(x) \, dx := |E|^{-1} \int_E f(x) \, dx$.

Lemma 6.4 *Let Ω, D, N be a standard Lipschitz graph domain for the mixed problem with Lipschitz constant M. There is an exponent $q_0 = q_0(M) > 2$ so that on any ball B with center in Ω, if u is an energy solution of*

$$\begin{cases} \Delta u = 0, & \text{in } 2B \cap \Omega \\ u = 0, & \text{in } D \cap 2B \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } N \cap 2B \end{cases}$$

then, for all $1 \leq q \leq q_0$, we have

$$\left(⨍_{B\cap\Omega} |\nabla u|^q \, dx \right)^{1/q} \leq \frac{C}{r} \left(⨍_{2B\cap\Omega} |u|^2 \, dx \right)^{1/2},$$

where r is the radius of B.

Proof. This follows from the Caccioppoli inequality and a reverse Hölder inequality as in Giaquinta [14]. ∎

The next estimate gives the decay at infinity of a solution to the mixed problem with atomic data.

Lemma 6.5 *Let Ω, D, N be a standard Lipschitz graph domain for the mixed problem (1.1). Let $\epsilon' > -1$ and $-1+2/q_0 < \epsilon < 1$, where q_0 is as in Lemma 6.4. If u is a solution of the mixed problem for $L^2(d\sigma_\epsilon)$ with zero Dirichlet data and with Neumann data a $\sigma_{\epsilon'}$-atom, a, which is supported in $\Delta_\rho(x_a)$, then we have*

$$\int_{\Sigma_k} |\nabla u|^2 \, d\sigma_\epsilon \leq C \frac{\sigma_\epsilon(\Sigma_k)}{(2^k\rho)^2} \left[\sup_{\cup_{|j|\leq 1} \tilde{R}_{k+j}} |u| \right]^2, \qquad \text{for all } k \geq 2\,.$$

Proof. With the same notations as in Lemma 6.3, we let $\eta_k \geq 0$ be a smooth cutoff function which equals one on R_k, is zero outside $\tilde{R}_k$, and satisfies $|\nabla \eta| \leq C/(2^k\rho)$. We let α denote a vector field of the form $\alpha(z) = (\mathrm{Re}(e^{i\lambda}z^\epsilon), \mathrm{Im}(e^{i\lambda}z^\epsilon))$, for some $\lambda \in \mathbf{R}$ and for ϵ as in the hypothesis.

We apply the Rellich identity with vector field $\alpha\eta_k$ where α is as in Lemma 4.4 to conclude that for $k \geq 2$, we have

$$\int_{\Sigma_k} |\nabla u|^2 \, d\sigma_\epsilon \leq \frac{C}{2^k\rho} \int_{\tilde{R}_k} |\nabla u|^2 |\alpha| dy. \tag{6.9}$$

(This uses the fact that the mixed data of u is zero on the support of η_k when $k \geq 2$).

Applying Hölder's inequality we obtain

$$\int_{\tilde{R}_k} |\nabla u|^2 |\alpha| \, dy \leq \left(\int_{\tilde{R}_k} |\nabla u|^{2p} \, dx \right)^{1/p} \left(\int_{\tilde{R}_k} |\alpha|^{p'} \, dy \right)^{1/p'}, \tag{6.10}$$

where $1/p + 1/p' = 1$ and p lies in the interval $1/(1-|\epsilon|) < p \leq q_0/2$ (if $-1+2/q_0 < \epsilon < 0$) or, if $\epsilon > 0$, $1 \leq p \leq q_0/2$. (These conditions grant that Lemmata 6.4, 6.3 and 3.1 apply in what follows). We now cover $\tilde{R}_k$ with a (fixed) number of balls $B_{k,n}$ (each centered at a point in Ω), $n = 1, ..., m$, such that $\mathrm{diam} B_{k,n} \approx 2^k\rho$ and

$$\tilde{R}_k \subseteq \cup_{n=1}^m B_{k,n} \subseteq \cup_{|j|\leq 2} R_{k+j} = \cup_{|j|\leq 1} \tilde{R}_{k+j}\,.$$

By Lemma 6.4, for p as above, we have

$$\left(\int_{\tilde{R}_k} |\nabla u|^{2p} \, dy \right)^{1/p} \leq C(2^k\rho)^{\frac{2}{p}-4} \sum_{|j|\leq 1} \int_{\tilde{R}_{k+j}} |u|^2 \, dy.$$

Moreover, Lemma 6.3 and Lemma 3.1, also for p as above, imply

$$\left(\int_{\tilde{R}_k} |\alpha|^{p'} \, dy \right)^{1/p'} \leq C(2^k\rho)^{1-\frac{2}{p}} \sigma_\epsilon(\Sigma_k). \tag{6.11}$$

Combining (6.9) to (6.11), we obtain

$$\begin{aligned}\int_{\Sigma_k} |\nabla u|^2 \, d\sigma_\epsilon &\leq \frac{C}{2^k\rho}(2^k\rho)^{\frac{2}{p}-4}(2^k\rho)^{1-\frac{2}{p}}\sigma_\epsilon(\Sigma_k)\sum_{|j|\leq 1}\int_{\tilde{R}_{k+j}} |u|^2 \, dx \\ &\leq C\sigma_\epsilon(\Sigma_k)(2^k\rho)^{-2}\left[\sup_{x\in\cup_{|j|\leq 1}\tilde{R}_{k+j}} |u|\right]^2 .\end{aligned}$$

∎

Lemma 6.6 *Let Ω, N, D be a standard Lipschitz graph domain for the mixed problem. Suppose u is the solution of the mixed problem for $L^2(d\sigma_\epsilon)$, see (1.1), with zero Dirichlet data and with Neumann data an $H^1(N, d\sigma_{\epsilon'})$-atom, a, supported in a surface ball $\Delta_\rho(x_a)$. Let $\epsilon' > -1$ and $-1 + 2/q_0 < \epsilon < 1$ (with q_0 as in Lemma 6.4). Then, for all integers $k \geq 5$ we have*

$$\int_{\Sigma_k} (\nabla u)^*(x)^2 \, d\sigma_\epsilon \leq \frac{C\sigma_\epsilon(\Sigma_k)}{2^{2k(1+\delta)}(\sigma_{\epsilon'}(\Delta_\rho(x_a)))^2},$$

where δ is as in (6.3).

Proof. By Lemmata 6.2 and 6.5, we have

$$\begin{aligned}\int_{\Sigma_k} |\nabla u|^2 \, d\sigma_\epsilon &\leq \frac{C}{(2^k\rho)^2}\sigma_\epsilon(\Sigma_k)\left(\frac{\rho^\delta}{(2^k\rho)^\delta}\frac{\sigma(\Delta_\rho(x_a))}{\sigma_{\epsilon'}(\Delta_\rho(x_a))}\right)^2 \\ &= \frac{C}{(2^k\rho)^2}\sigma_\epsilon(\Sigma_k)2^{-2k\delta}\left(\frac{\rho}{\sigma_{\epsilon'}(\Delta_\rho(x_a))}\right)^2 . \qquad (6.12)\end{aligned}$$

In order to pass from the estimate above for ∇u to an estimate for $(\nabla u)^*$, we use the Cauchy kernel to represent ∂u, the complex derivative of u, see (4.6). To carry out this argument, we consider a cutoff function η_k that is supported in $B_{2^{k+4}\rho}(x_a) \setminus B_{2^{k-3}\rho}(x_a)$, equals one on $B_{2^{k+3}\rho}(x_a) \setminus B_{2^{k-2}\rho}(x_a)$ and, furthermore, satisfies: $|\partial\eta_k| \leq C/2^k\rho$. On account of the analyticity of ∂u, the Cauchy-Pompeiu formula, see e.g. Bell [2], yields

$$\eta_k(z)\partial u(z) = K_{\partial\Omega}(\eta_k\partial u)(z) + I(z), \quad z \in \Omega, \qquad (6.13)$$

where we have set

$$K_{\partial\Omega}(\eta_k\partial u)(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{1}{z-\zeta}\eta_k(\zeta)\partial u(\zeta) \, d\zeta , \quad z \in \Omega, \qquad (6.14)$$

and $d\zeta$ is used to denote complex line integration. Also, we define

$$I(z) = \frac{1}{\pi}\int_{R_{k-2}\cup R_{k+3}}\frac{1}{z-y}\bar{\partial}\eta_k(y)\partial u(y) \, dy \qquad (6.15)$$

and in this expression, dy denotes area measure.

Next, for $x \in \Sigma_k$ we decompose the sector $\Gamma(x) = \Gamma^n(x) \cup \Gamma^f(x)$ where $\Gamma^n(x) = \Gamma(x) \cap B_{\kappa 2^k \rho}(x_a)$, $\Gamma^f(x) = \Gamma(x) \setminus B_{\kappa 2^k \rho}(x_a)$, and the constant κ is chosen so that $\Gamma^n(x) \subset \cup_{|j|\le 1} R_{k+j}$ for $x \in \Sigma_k$. If we let $v_n^*(x)$ denote the supremum of $|v|$ on $\Gamma_n(x)$ and similarly for v_f^*, using (6.13) and the theorem of Coifman, McIntosh and Meyer [7] we obtain

$$\begin{aligned}\int_{\Sigma_k} (\nabla u)_n^*(x)^2\, d\sigma_\epsilon(x) &= \int_{\Sigma_k} (\partial u)_n^*(x)^2\, d\sigma_\epsilon(x) \qquad (6.16)\\ &\le \sum_{|j|\le 3} \int_{\Sigma_{k+j}} |\partial u(x)|^2\, d\sigma_\epsilon(x)\\ &\qquad + \sigma_\epsilon(\Sigma_k)\left[\sup_{x\in\Sigma_k}\left(\sup_{z\in \cup_{|j|\le 1} R_{k+j}} |I(z)|\right)^2\right],\end{aligned}$$

where we have used that $\operatorname{supp} \eta_k \subset \cup_{|j|\le 3} R_{k+j}$.

We now estimate the term $I(z)$: by the Cauchy-Schwartz inequality we have

$$|I(z)| \le \frac{C}{\rho 2^{k-1}} \Big(\int_{R_{k-3}\cup R_{k+3}} |\nabla u|^2\, dy\Big)^{\frac{1}{2}}, \quad z \in \cup_{|j|\le 1} R_{k+j}\,.$$

On account of the vanishing boundary conditions on u and $\partial u/\partial \nu$ we may now apply Caccioppoli inequality and conclude

$$I(z) \le \frac{1}{2^k \rho} \sup_{y \in \cup_{|j|\le 4} R_{k+j}} |u(y)| \quad \text{for all } z \in \cup_{|j|\le 1} R_{k+j} \quad \text{and } x \in \Sigma_k\,.$$

Using interior estimates, we also obtain

$$(\nabla u)_f^*(x) \le \frac{1}{2^k \rho} \sup_{z \in \Omega \setminus B_{2^{k-1}\rho}} |u(z)|\,, \quad x \in \Sigma_k\,.$$

The conclusion now follows from (6.12), (6.16) and Lemma 6.2. ∎

We are ready to prove Theorem 6.1.

Proof of Theorem 6.1. We fix $\epsilon > 0$ such that $2\beta/(\pi - 2\beta) < \epsilon < 1$ so that, for ϵ' as in the hypotheses we have: $(\epsilon - 1)/2 < \epsilon'$. The proof is based on the following elementary observations: given any $\epsilon' > -1$ and $\epsilon > -1$, if a is a $\sigma_{\epsilon'}$-atom then a lies in $L^2(N, d\sigma_\epsilon)$; by Theorem 4.6 we may solve the mixed problem (1.1) for $L^2(d\sigma_\epsilon)$ with Neumann data a and zero Dirichlet data (provided β and ϵ are as in the hypothesis). We let u denote such a solution and show that u satisfies (6.1) and (6.2). We begin by studying ∇u near the support of a. By the Cauchy-Schwarz inequality, together with the normalization of a and the estimate for the $L^2(d\sigma_\epsilon)$-mixed problem (see Theorem 4.6), we have

$$\begin{aligned}\int_{\Delta_{2^{10}\rho}(x_a)} (\nabla u)^*(x)\, d\sigma_{\epsilon'} &\leq \left(\int_{\Delta_{2^{10}\rho}(x_a)} (\nabla u)^*(x)^2\, d\sigma_\epsilon\right)^{1/2} \sigma_{2\epsilon'-\epsilon}(\Delta_{2^{10}\rho}(x_a))^{1/2} \\ &\leq C\|a\|_{L^2(\sigma_\epsilon)}\sigma_{2\epsilon'-\epsilon}(\Delta_{2^{10}\rho}(x_a))^{1/2} \\ &\leq C\frac{\sigma_\epsilon(\Delta_\rho(x_a))^{1/2}\sigma_{2\epsilon'-\epsilon}(\Delta_{2^{10}\rho}(x_a))^{1/2}}{\sigma_{\epsilon'}(\Delta_\rho(x_a))},\end{aligned}$$

provided $2\epsilon' - \epsilon > -1$, that is $\epsilon' > (\epsilon - 1)/2$ (note that the latter is bounded below by $(4\beta - \pi)/2(\pi - 2\beta) > -1$). Lemma 3.1 now grants

$$\frac{\sigma_\epsilon(\Delta_\rho(x_a))^{1/2}\sigma_{2\epsilon'-\epsilon}(\Delta_{2^{10}\rho}(x_a))^{1/2}}{\sigma_{\epsilon'}(\Delta_\rho(x_a))} \leq C.$$

Next, we consider ∇u away from the support of a: we will show that there is $\eta > 0$ so that

$$\int_{\Sigma_k} (\nabla u)^*(x)\, d\sigma_{\epsilon'} \leq C2^{-\eta k}. \tag{6.17}$$

Summing over k will give the estimate for $(\nabla u)^*$.

We begin with the Cauchy-Schwarz inequality and then use the estimate of Lemma 6.6 (note that we have $\epsilon > 2\beta/(\pi - 2\beta) > 0 > -1 + 2/q_0$) to obtain

$$\begin{aligned}\int_{\Sigma_k} (\nabla u)^*(x)\, d\sigma_{\epsilon'}(x) &\leq \left(\int_{\Sigma_k} (\nabla u)^*(x)^2\, d\sigma_\epsilon(x)\right)^{1/2} \sigma_{2\epsilon'-\epsilon}(\Sigma_k)^{1/2} \\ &\leq C\frac{\sigma_{2\epsilon'-\epsilon}(\Sigma_k)^{1/2}\sigma_\epsilon(\Sigma_k)^{1/2}}{\sigma_{\epsilon'}(\Delta_\rho(x_a))2^{k(1+\delta)}} \\ &= C2^{-\delta k}\frac{\max(|x_a|, 2^k\rho)^{\epsilon'}}{\max(|x_a|, \rho)^{\epsilon'}}\end{aligned}$$

where the last inequality follows from Lemma 3.1. We have

$$\frac{\max(|x_a|, 2^k\rho)}{\max(|x_a|, \rho)} = \begin{cases} 1, & \frac{|x_a|}{\rho} < 1 \\ |x_a|/\rho, & 1 < \frac{|x_a|}{\rho} < 2^k \\ 2^k, & \frac{|x_a|}{\rho} > 2^k. \end{cases} \tag{6.18}$$

In particular, we have

$$1 \leq \frac{\max(|x_a|, 2^k\rho)}{\max(|x_a|, \rho)} \leq 2^k.$$

Thus, letting $\eta := \delta - \max\{\epsilon', 0\} > 0$, we obtain (6.17). Summing over k yields estimate (6.1).

Now we indicate why $\partial u/\partial\nu$ lies in $H^1(\partial\Omega, d\sigma_{\epsilon'})$. A similar, but simpler, argument shows that u lies in $H^{1,1}(\partial\Omega, d\sigma_{\epsilon'})$. We first prove the vanishing moment condition:

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\sigma = 0. \tag{6.19}$$

To this end, we observe that, since $\int_{\partial\Omega} a\, d\sigma = 0$, we may proceed as in the proof of Lemma 6.5 and inequality (6.17) to obtain

$$\int_{\partial\Omega} (\nabla u)^*\, d\sigma < +\infty. \tag{6.20}$$

On account of (6.20) we may now apply the divergence theorem to obtain (6.19). To continue, we follow the arguments of Coifman and Weiss [8]. We begin by writing

$$\frac{\partial u}{\partial \nu} = \sum_{k=0}^{\infty} b_k$$

where $b_0 := \chi_{\Delta_0}(\frac{\partial u}{\partial \nu} - ⨍_{\Delta_\rho} \frac{\partial u}{\partial \nu} d\sigma)$ and, for $k \geq 1$, $b_k := \chi_{\Sigma_k}\frac{\partial u}{\partial \nu} + \chi_{\Delta_{k-1}} ⨍_{\Delta_{k-1}} \frac{\partial u}{\partial \nu} d\sigma - \chi_{\Delta_k} ⨍_{\Delta_k} \frac{\partial u}{\partial \nu} d\sigma$. The estimate of Lemma 6.6 and arguments used above imply that

$$\int_{\partial\Omega} |b_k|\, d\sigma_{\epsilon'} \leq \left(\int_{\Delta_k} |b_k|^2\, d\sigma_\epsilon \right)^{1/2} \sigma_{2\epsilon'-\epsilon}(\Delta_k)^{1/2} \leq 2^{-\eta k}.$$

where again $\eta := \delta - \max(\epsilon', 0) > 0$. The one tricky point in this argument is that we must use (6.19) to obtain that $\int_{\Delta_k} \frac{\partial u}{\partial \nu}\, d\sigma = -\int_{\partial\Omega\setminus\Delta_k} \frac{\partial u}{\partial \nu}\, d\sigma$. Thus we have that $2^{\eta k} b_k$ is normalized in $L^1(d\sigma_{\epsilon'})$. In order to prove that b_k is in $H^1(d\sigma_{\epsilon'})$, one now proceeds as in Strömberg and Torchinsky [32, Theorem 1, p. 111] to show that the grand maximal function is in $L^1(d\sigma_{\epsilon'})$ and then find an atomic decomposition. This argument gives estimate (6.2) for $\partial u/\partial \nu$ in $H^1(d\sigma_{\epsilon'})$; the corresponding estimate for u is obtained in a similar fashion. ∎

7 Existence and uniqueness for H^1 and L^p

In this section, we give the final arguments to prove our main result, Theorem 1.1 existence and uniqueness for the mixed problem.

In Section 4 we obtained existence and uniqueness for the solution of the $L^2(d\sigma_\epsilon)$-mixed problem for ϵ in an interval which includes positive values of ϵ and does not include 0. In this section, we will study solutions which have atomic data and show that the non-tangential maximal function for such solutions will lie in $L^1(d\sigma_{\epsilon'})$ for ϵ' small.

Our results are restricted to domains with Lipschitz constant M that is less than 1. This restriction is inherited from the previous sections (Lemma 4.4). We do not make an effort to find the largest value of p_0 (nor do we expect that the restriction $M < 1$ is essential). However, as the example discussed in the introduction indicates, we cannot expect to always have $p_0 \geq 2$, even in the case of smooth domains. Furthermore, in Brown [4] the mixed problem is solved in $L^2(d\sigma)$ for certain Lipschitz domains with arbitrarily large Lipschitz constant; thus, it is more than the Lipschitz constant that governs the solvability of this problem.

We begin with our uniqueness result.

Lemma 7.1 *Let $\delta > 0$ be as in Lemma 6.2. Suppose that ϵ' and $p \geq 1$ satisfy: $-\delta < \epsilon' \leq 0$, $1/p' - \epsilon'/p < \delta$. Under these hypotheses, the solution of the mixed problem (1.1) for $L^p(d\sigma_{\epsilon'})$ is unique.*

Proof. Suppose that u solves the mixed problem for $L^p(d\sigma_{\epsilon'})$ with zero data, that is

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } D \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } N \\ (\nabla u)^* \in L^p(d\sigma_{\epsilon'}) \end{cases} \tag{7.1}$$

We will show that u solves the regularity problem (2.5) with zero data and then use Lemma 2.6 to conclude that $u = 0$. To see that u vanishes on $N \subset \partial\Omega$, we will show that there is $\eta > 0$ such that for any $H^1(d\sigma_\eta)$-atom, a, we have

$$\int_N u\, a\, d\sigma = 0\,. \tag{7.2}$$

This means that u is constant almost everywhere. In order to prove (7.2), we fix $\eta > 0$ so that

$$0 < \frac{1}{p'} - \frac{\epsilon'}{p} < \eta < \delta$$

(where $1/p + 1/p' = 1$) and we let a be a σ_η-atom supported in a boundary ball $\Delta_r(x_a)$. According to (the proof of) Theorem 6.1, the mixed problem in $H^1(d\sigma_\eta)$ with Neumann data a and with zero Dirichlet data has a solution v that satisfies

$$|v(z)| \leq C|z - x_a|^{-\delta}, \quad \text{if } \ |z - x_a| \geq 2\rho\,; \tag{7.3}$$

moreover, for ϵ as in Theorem 4.6, we have

$$(\nabla v)^* \in L^2(d\sigma_\epsilon) \cap L^1(d\sigma_\eta). \tag{7.4}$$

We will need a pointwise estimate for u; to this end, given z in Ω, we consider the path in Ω from 0 to z given by: $\gamma_z(t) = (tz_1, \phi(tz_1) + t(z_2 - \phi(z_1)))$. Recall that ϕ is the function whose graph gives $\partial\Omega$. We define the following Carleson measure with respect to $d\sigma$:

$$d\mu_z(y) := \chi_{B_{2|z|}}(y)\, d\mathcal{H}_1|_{\gamma_z}\,, \quad y \in \Omega\,,$$

where $\mathcal{H}_1$ denotes 1−dimensional Hausdorff measure. By the Fundamental Theorem of Calculus, properties of Carleson measures (Proposition 2.2) and Hölder inequality, we have

$$\begin{aligned}
|u(z)| &\leq \int_{\gamma_z} |\nabla u|\,|d\gamma_z| \\
&\leq C\int_{\Delta_{2|z|}(0)} (\nabla u)^* d\sigma \\
&\leq |z|^{\frac{1}{p'}-\frac{\epsilon'}{p}} \|(\nabla u)^*\|_{L^p(d\sigma_{\epsilon'})}. \qquad (7.5)
\end{aligned}$$

Note that by a similar argument we may show that u is locally bounded on $\partial\Omega$.

With these estimates collected, we now proceed to the main part of the argument. Let R be large and let ψ_R be a cutoff function which is equal to 1 on $B_R(0)$, zero outside $B_{2R}(0)$ and such that $|\nabla\psi_R| \leq C/R$. We apply Green's second identity to the pointwise products $v\,\psi_R$ and $u\,\psi_R$ and obtain

$$\int_{\partial\Omega} \psi_R^2 \left(v\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu}\right) d\sigma = 2\int_\Omega \psi_R(v\nabla u\cdot\nabla\psi_R - u\nabla v\cdot\nabla\psi_R)\,dy. \qquad (7.6)$$

Concerning the left-hand side of (7.6), on account of the boundary conditions satisfied by u and v, we have

$$\int_{\partial\Omega} \psi_R^2 \left(v\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu}\right) d\sigma = -\int_N \psi_R^2\, u\, a\, d\sigma\,.$$

Since a is compactly supported and u is locally bounded (see (7.5) and comment thereafter) we may apply the Lebesgue dominated convergence theorem and conclude

$$\lim_{R\to\infty}\int_{\partial\Omega} \psi_R^2 \left(v\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu}\right) d\sigma = -\int_N u\, a\, d\sigma\,. \qquad (7.7)$$

Concerning the right-hand side of (7.6), we will show that

$$\lim_{R\to\infty}\int_\Omega \psi_R\, v\nabla u\cdot\nabla\psi_R\,dy \;=\; \lim_{R\to\infty}\int_\Omega \psi_R\, u\nabla v\cdot\nabla\psi_R\,dy \;=\; 0. \qquad (7.8)$$

Indeed, if R is large, then from Lemma 6.2, for z in the support of $\nabla\psi_R$, we have

$$|v(z)| \leq CR^{-\delta} \qquad (7.9)$$

and it follows that

$$\int_\Omega \psi_R\, v\nabla u\cdot\nabla\psi_R\,dy \leq CR^{-\delta}\int_\Omega |\nabla\psi_R||\nabla u|\,dy.$$

By Hölder inequality we have

$$\begin{aligned}
\int_\Omega |\nabla\psi_R||\nabla u|\,dy &\leq R^{-\frac{\epsilon'}{p}}\int_{B_R(0)} |\nabla\psi_R|\,|\nabla u|\,|y|^{\frac{\epsilon'}{p}}\,dy \\
&\leq R^{\frac{1}{p'}-\frac{\epsilon'}{p}}\left(\int_\Omega |\nabla u|^p\,\chi_{B_R(0)}\,\frac{|y|^{\epsilon'}}{R}dy\right)^{\frac{1}{p}}.
\end{aligned}$$

By Carleson Theorem, see (2.3) and Proposition 2.2, the last term in the inequalities above is bounded by

$$CR^{\frac{1}{p'}-\frac{\epsilon'}{p}}\|(\nabla u)^*\|_{L^p(d\sigma_{\epsilon'})}\,.$$

We conclude

$$\left|\int_\Omega \psi_R(v\nabla u\cdot\nabla\psi_R)dy\right| \le CR^{\frac{1}{p'}-\frac{\epsilon'}{p}-\delta}\|(\nabla u)^*\|_{L^p(d\sigma_{\epsilon'})}; \tag{7.10}$$

our hypotheses on η, ϵ' and p imply that the exponent of R is negative, so the first integral in (7.8) vanishes as $R\to\infty$. To handle the second integral in (7.8) we apply (7.5); using the fact the $\nabla\psi_R$ is supported in the annulus $R<|z|<2R$ we obtain

$$\left|\int_\Omega \psi_R\, u\nabla\psi_R\cdot\nabla v\, dy\right| \le CR^{\frac{1}{p'}-\frac{\epsilon'}{p}}\|(\nabla u)^*\|_{L^p(\partial\Omega.d\sigma_{\epsilon'})}R^{-\eta}\int_{R<|y|<2R}|\nabla v|\frac{|y|^\eta}{R}dy.$$

By applying Proposition 2.2 on Carleson measures one more time, we see that the latter is bounded by

$$CR^{\frac{1}{p'}-\frac{\epsilon'}{p}-\eta}\|(\nabla u)^*\|_{L^p(d\sigma_{\epsilon'})}\int_{\partial\Omega}(\nabla v)^*\,d\sigma_\eta.$$

It follows that the second integral in (7.8) also vanishes as R tends to infinity. This completes the proof of (7.8) and of this Lemma. ∎

Theorem 7.2 *Suppose Ω, N, D is a standard Lipschitz graph domain for the mixed problem, with Lipschitz constant $M<1$ and set $\beta=\arctan(M)$. There is $\delta=\delta(M)$ satisfying $0<\delta<1$ so that, for $\max\{-\delta\,,\frac{4\beta-\pi}{2(\pi-2\beta)}\}<\epsilon'\le 0$ the mixed problem (1.1) for $L^1(d\sigma_{\epsilon'})$ with Dirichlet data in $H^{1,1}(D,d\sigma_{\epsilon'})$ and with Neumann data in $H^1(N,d\sigma_{\epsilon'})$ is uniquely solvable. The solution u satisfies the following estimates*

$$\int_{\partial\Omega}(\nabla u)^*(x)\,d\sigma_{\epsilon'} \;\le\; C(M,\epsilon')\left(\|f_N\|_{H^1(N,d\sigma_{\epsilon'})}+\|f_D\|_{H^{1,1}(D,d\sigma_{\epsilon'})}\right); \tag{7.11}$$

$$\|u\|_{H^{1,1}(d\sigma_{\epsilon'})}+\left\|\frac{\partial u}{\partial\nu}\right\|_{H^1(d\sigma_{\epsilon'})} \le C(M,\epsilon')\left(\|f_N\|_{H^1(N,d\sigma_{\epsilon'})}+\|f_D\|_{H^{1,1}(D,d\sigma_{\epsilon'})}\right). \tag{7.12}$$

Proof. We first observe that we may use the solution of the regularity problem from Theorem 5.1 to reduce to the case where the Dirichlet data is zero. More precisely, we consider the (unique) solution v of the regularity problem (2.5) with data $\tilde f_D\in H^{1,1}(d\sigma_{\epsilon'})$ (here $\tilde f_D$ denotes an extension of f_D to $\partial\Omega$). By Theorem 5.1 it follows that $\partial v/\partial\nu\in H^1(\partial\Omega,d\sigma_{\epsilon'})$, and it is easy to see that u is the unique solution of the mixed problem with data f_D and f_N if and only if $u-v$ solves the mixed problem with zero Dirichlet data and with Neumann data $g_N:=f_N-\partial v/\partial\nu$.

To prove existence we consider any atomic decomposition for g_N, namely

$$g_N(x)=\sum_{j=1}^\infty\lambda_j a_j(x),\quad \sum_{j=1}^\infty|\lambda_j|<+\infty\,, \tag{7.13}$$

see (5.1) and (5.2). By Theorem 6.1 it follows that for each j the mixed problem:

$$\begin{cases} \Delta h_j = 0 & \text{in } \Omega \\ h_j = 0 & \text{on } D \\ \frac{\partial h_j}{\partial \nu} = a_j & \text{on } N \\ (\nabla h_j)^* \in L^1(d\sigma_{\epsilon'}) \end{cases} \tag{7.14}$$

has a solution h_j that satisfies:

$$\|h_j\|_{H^{1,1}(d\sigma_{\epsilon'})} + \left\| \frac{\partial h_j}{\partial \nu} \right\|_{H^1(d\sigma_{\epsilon'})} \leq C(M, \epsilon'). \tag{7.15}$$

Thus, the function

$$h := \sum_{j=1}^{\infty} \lambda_j h_j$$

is a solution of the mixed problem with zero Dirichlet data and with Neumann data g_N, and it satisfies:

$$\left\| \frac{\partial h}{\partial \nu} \right\|_{H^1(d\sigma_{\epsilon'})} \leq C(M, \epsilon') \sum_{j=1}^{\infty} |\lambda_j| \, .$$

By Lemma 7.1, h is unique, (i.e. h is independent of the choice of the atomic decomposition for g_N); taking the infimum over all atomic decompositions of g_N now yields (7.11). This proves existence; uniqueness follows from Lemma 7.1. ∎

Next, we recall a few well known results concerning the the complex interpolation spaces of weighted L^p and Hardy spaces, see Bergh and Löfstrom [3, Theorem 5.5.3, Corollary 5.5.4], Strömberg and Torchinsky [32, Theorem 3, pg. 179]. For weights w_0 and w_1, and exponents p_0, p_1 we set

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad w_\theta = w_0^{\frac{(1-\theta)p_\theta}{p_0}} w_1^{\frac{\theta p_\theta}{p_1}}, \quad 0 \leq \theta \leq 1. \tag{7.16}$$

We let $[A, B]_\theta$ denote the complex interpolation space of index θ as defined in the monograph of Bergh and Löfstrom [3, Chapt. 4].

Theorem 7.3 *Suppose w_0 and w_1 are weights, then we have*

$$[L^1(w_0 \, d\sigma), L^2(w_1 \, d\sigma)]_\theta = L^{p_\theta}(w_\theta \, d\sigma).$$

If in addition the weights w_j are in $A_\infty(d\sigma)$ for $j = 0, 1$, then we have

$$[H^1(w_0 \, d\sigma), L^2(w_1 \, d\sigma)]_\theta = L^{p_\theta}(w_\theta \, d\sigma).$$

Moreover, if a linear operator T is bounded:

$$\begin{aligned} T : H^1(w_0 \, d\sigma) &\rightarrow L^1(w_0 \, d\sigma), \\ T : L^2(w_1 \, d\sigma) &\rightarrow L^2(w_1 \, d\sigma) \end{aligned}$$

with norms M_0 and M_1 respectively, then T is bounded:

$$L^{p_\theta}(w_\theta\, d\sigma) \to L^{p_\theta}(w_\theta\, d\sigma)$$

with norm M satisfying

$$M \leq C M_0^{1-\theta} M_1^{\theta}.$$

Remark. The constant C in the estimate for the operator norm is 1 when we consider Lebesgue spaces. It may not be one for Hardy spaces, see Strömberg and Torchinsky [32].

We will focus on the case: $w_0\, d\sigma = d\sigma_{\epsilon'}$, with $\epsilon' < 0$ as in Theorem 7.2, and $w_1\, d\sigma = d\sigma_\epsilon$, where $\epsilon > 0$ is as in Theorem 4.6. We are now ready to give the proof of our main result, Theorem 1.1.

Proof of Theorem 1.1. We first use a result of Dahlberg and Kenig [10, Theorem 3.8] to reduce to the case where the Dirichlet data in the mixed problem is zero. (While Dahlberg and Kenig only discuss $n \geq 3$ in their work, one can extend their results to two dimensions.)

We will use interpolation to establish existence of solutions satisfying the estimate (1.4). Because the complex method applies to linear operators, we employ a standard technique to obtain the non-tangential maximal function as a supremum of linear operators. Fix $\{y_j\}_{j\in\mathbf{N}}$, a dense subset of the sector $\Gamma(0)$ and let $\mathcal{E} = \{E_j\}$ be decomposition of $\partial\Omega$ into disjoint measurable subsets, E_j. We define a linear operator:

$$f_N \in H^1(d\sigma_{\epsilon'}) + L^2(d\sigma_\epsilon) \;\to\; T_{\mathcal{E}}(f_N)(x) := \sum_j \chi_{E_j}(x)\nabla u(x+y_j)\,, \quad x \in \partial\Omega,$$

where u is the solution of the $H^1(d\sigma_{\epsilon'})$-mixed problem (resp. the $L^2(d\sigma_\epsilon)$-mixed problem) for data f_N and $f_D = 0$. Note that for a suitable sequence of decompositions $\{\mathcal{E}_k\}$, we have

$$(\nabla u)^*(x) = \lim_{k\to\infty} |T_{\mathcal{E}_k} f_N(x)| \quad a.e.\ x \in \partial\Omega\,.$$

Now complex interpolation, (see Theorem 7.3) implies that the operator $T_{\mathcal{E}}$ is bounded on the intermediate spaces with a norm independent of $\mathcal{E}$. Fatou's lemma then yields boundedness for the non-tangential maximal function. It is easy to see that for the spaces $L^2(d\sigma_\epsilon)$, $(2\beta)/(\pi-2\beta) < \epsilon < 1$ and $H^1(d\sigma_{\epsilon'})$, $\max(-\delta, (\epsilon-1)/2) < \epsilon' \leq 0$, the intermediate spaces include $L^p(d\sigma)$ for $1 < p < p_1(M)$ where

$$p_1(M) = \frac{2(\pi-2\beta)\min(\delta, \frac{\pi-4\beta}{2(\pi-2\beta)}) + 2\beta}{(\pi-2\beta)\min(\delta, \frac{\pi-4\beta}{2(\pi-2\beta)}) + 2\beta}.$$

In addition, in Lemma 7.1, uniqueness was established for p in the range: $1 < p < p_2(M)$, $p_2(M) = 1/(1-\delta(M))$. Thus, we have existence and uniqueness for $p \in (1, p_0(M))$ with $p_0 = \min(p_1(M), p_2(M))$. ∎

8 Final remarks

We close by listing a few open questions.

- Can we remove the restriction that the Lipschitz constant of the domain is at most one?
- Can we obtain similar results in higher dimensions? The obvious problem here is that our two-dimensional weighted estimates rely on a Rellich identity which is based on complex function theory.
- Can we study domains where the boundary between D and N is more interesting? For example in $\mathbf{R}^3$, let $\Omega = \{x : x_3 > c(|x_1| + |x_2|)\}$ and let $D = \partial\Omega \cap \{x : x_1 x_2 > 0\}$ and then $N = \partial\Omega \setminus D$. Can we solve the mixed problem for some L^p space in this domain?
- Can we extend the solution of the mixed problem to general domains, rather than only graph domains?

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