The Mixed Boundary Problem in L^p and Hardy spaces for Laplace's Equation on a Lipschitz Domain

Jeffery D. Sykes and Russell M. Brown

Abstract. We study the boundary regularity of solutions of the mixed problem for Laplace's equation in a Lipschitz graph domain Ω whose boundary is decomposed as $\partial\Omega = N \cup D$, where $N \cap D = \emptyset$. For a subclass of these domains, we show that if the Neumann data g is in $L^p(N)$ and if the Dirichlet data f is in the Sobolev space $L^{p,1}(D)$, for $1 < p < 2$, then the mixed boundary problem has a unique solution u for which $N(\nabla u) \in L^p(\partial\Omega)$, where $N(\nabla u)$ is the non-tangential maximal function of the gradient of u.

1. Introduction

In this paper, we consider the mixed boundary value problem for Laplace's equation in a domain $\Omega \subset \mathbf{R}^n$, $n \geq 3$. We assume that the boundary of Ω is decomposed as $\partial\Omega = N \cup D$, where $N \cap D = \emptyset$. The mixed problem is stated as follows. Given functions f_N and f_D, find a function u which satisfies

$$(1) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f_D & \text{on } D, \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N, \end{cases}$$

where $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ represents the outer normal derivative on $\partial\Omega$. Problems of this kind, including those with elliptic operators more general than the Laplacian and with first order data more general than the normal derivative, have been studied for a long time from many points of view. It is well known that solutions exist under mild conditions on the domain and the data. However, it is also known that the solutions of the more general mixed problems are not generally smooth, regardless of how regular the data may be. In particular, problems arise in a neighborhood of the boundary between N and D. The typical counterexample is given by the harmonic function

$$u(x, y) = \mathrm{Im}(x + iy)^{1/2}\ , \ \ x \in \mathbf{R}\,,\ y > 0,$$

which is zero on the positive real line and whose normal derivative is zero on the negative real line.

1991 *Mathematics Subject Classification.* Primary 35J25.

Both authors received partial support from the National Science Foundation, Division of Mathematical Sciences.

In results dealing with classical solutions, much work has focused on the conditions necessary to obtain more regular solutions. In [**15**] and [**16**], Lieberman addresses the Hölder continuity of classical solutions of the mixed problem with smooth data, proving optimal regularity results under certain conditions requiring compatibility between the elliptic operator and the boundary decomposition.

As another example, Azzam and Kreyszig [**1**] consider the mixed problem for elliptic equations in a plane domain with corners. Assuming that the Dirichlet data has Hölder continuous second derivatives and the remaining data has Hölder continuous first derivatives, they prove that the solution has Hölder continuous second derivatives, provided certain conditions are met which relate the interior angle at a corner with the coefficients of the operator at that corner. Their paper also provides a good history of the mixed problem on both smooth domains and domains with corners on the boundary.

Much effort has also been dedicated to the study of the mixed problem in polygonal domains. In his monograph, Grisvard [**12**] addresses both generalized solutions (Chapter 4) and classical solutions (Chapter 6), proving existence and regularity for a larger class of spaces than we consider in our results. This is to be expected, however, as we consider a larger class of domains.

A more recent result is that of Giuseppe Savaré [**19**] who considers the mixed problem in smooth ($C^{1,1}$) domains and who shows that with sufficiently regular data, the solution lies in the Besov space $B^{2,3/2}_{\infty}(\Omega)$. This addresses one of the limiting cases of Brown's paper [**3**] where it is shown that if we consider the creased domains defined below, then we obtain a non-tangential maximal function estimate which implies that the solution lies in $B^{2,3/2}_2(\Omega)$. The argument connecting the nontangential function estimates and the Besov space results requires the square function studied by Dahlberg [**7**] and the arguments in Fabes's paper [**10**]. Strictly speaking, there is no overlap between the results of Brown and Savaré since no smooth domain satisfies the hypotheses in Brown's results and Savaré requires additional smoothness hypotheses on the data and the domain, compared to the results of Brown.

Since we are considering harmonic functions, our solutions are smooth in the interior. Our interest is in obtaining regularity on the boundary. In particular, we consider a specific class of Lipschitz domains and show that the solution has $\nabla u \in L^p(\partial\Omega)$ when the data f_N is in $L^p(N)$ and f_D is in the Sobolev space $L^{p,1}(D)$, $1 < p < 2$. This result requires, roughly speaking, that D and N meet at an angle which is strictly less than π. See Section 2 for the precise hypotheses.

Our estimates for solutions of the mixed problem will involve the nontangential maximal function. For a function w defined on Ω, the nontangential maximal function of w, denoted $N(w)$, is defined for $y \in \partial\Omega$ by

$$N(w)(y) = \sup_{x \in \Gamma_\alpha(y)} |w(x)|,$$

where for $\alpha > 0$, the nontangential approach region $\Gamma_\alpha(y)$ is defined as

$$\Gamma_\alpha(y) = \{x \in \Omega : |x - y| < (1 + \alpha)\delta(x)\}.$$

In this definition and below, we are using $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ to denote the distance to the boundary. We note that the dependence of $N(w)$ on α is not significant. In particular, nontangential maximal functions defined using different values of α have comparable L^p-norms (see [**24**], page 367, for example). For the purposes of this discussion, we require that, in the graph domains defined below, α be chosen so that $y+te_n \in \Gamma_\alpha(y)$ for any $t>0$. Here e_n is the unit vector in the x_n direction.

Boundary values are defined in the nontangential sense: we say that $u=f$ on $\partial\Omega$ if for a.e. $y \in \partial\Omega$,

$$\lim_{\substack{x\to y\\ x\in\Gamma_\alpha(y)}} u(x) = f(y).$$

Except for a set of measure zero, the existence of this limit is independent of the cone opening α. By the outer normal derivative $\frac{\partial u}{\partial \nu}(y)$, we mean $\nu(y)\cdot\nabla u(y)$, where $\nabla u(y)$ is defined nontangentially as above. If Ω is a Lipschitz graph domain and u is harmonic on Ω with $N(u)<\infty$ a.e. on $\partial\Omega$, then u is known to have nontangential limits a.e. on $\partial\Omega$ [**9**].

Our goal is to prove an L^p estimate for solutions of (1):

$$\|N(\nabla u)\|_{L^p(\partial\Omega)} \le C\left(\|f_D\|_{L^{p,1}(D)} + \|f_N\|_{L^p(N)}\right).$$

For $1<p<2$, this first appeared in the Ph.D. thesis of Jeffery Sykes [**23**]. For $p=2$, this result is due to Brown [**3**]. His methods are similar to those used by Jerison and Kenig [**13**] for the Dirichlet and Neumann problems, but Brown makes a new application of the Rellich identity. In particular, he uses a smooth vector-field α so that $|\alpha\cdot\nu| \ge \delta > 0$ a.e. on $\partial\Omega$, but $\alpha\cdot\nu$ changes sign as we cross the boundary between D and N. This can only happen when the normal is discontinuous, which helps explain our hypotheses on $\partial\Omega$, N, and D. This work provides a partial answer to problem 3.2.15 in Kenig's CBMS Lecture Notes [**14**].

In Section 2, we define the class of domains under consideration. The remainder of Section 2 concentrates on proving an L^1 estimate for the L^2 nontangential solution when the Neumann data is an atom for a Hardy space and the Dirichlet data is zero. To accomplish this, we mimic the work of Dahlberg and Kenig [**8**], reflecting the solution in Ω to obtain a solution in a neighborhood of infinity of an elliptic divergence form operator. This allows us to use the asymptotic expansion of Serrin and Weinberger [**20**] to prove that the solution decays at infinity faster than a fundamental solution. This and the same use of the Rellich identity as mentioned above then leads to the atomic estimate. In Section 3, we prove the uniqueness of solutions to the mixed problem with $N(\nabla u) \in L^p(\partial\Omega)$. This is proven by duality, using solutions to the mixed problem with nice data. In Section 4, we give the final steps of the proof of the L^p-result.

We follow the standard convention that C is a constant which varies. Since we are working on graph domains, the constants will depend only on the dimension and the number M which appears below.

2. Atomic Estimates

We begin by defining our problem more precisely. For any point $x \in \mathbf{R}^n$, $n \ge 3$, we denote $x=(x',x_n)=(x_1,x'',x_n)$, where $x_1, x_n \in \mathbf{R}$, $x'' \in \mathbf{R}^{n-2}$, and $x'=(x_1,x'') \in \mathbf{R}^{n-1}$. Let $\Omega = \{x : x_n > \phi(x')\}$ be a Lipschitz graph domain, and let $\psi : \mathbf{R}^{n-2} \to \mathbf{R}$ be a Lipschitz function. Define

$$\begin{array}{rcl} N & = & \{x : x_1 \ge \psi(x'')\} \cap \partial\Omega \\ D & = & \{x : x_1 < \psi(x'')\} \cap \partial\Omega. \end{array} \tag{2}$$

We assume there is a constant M so that ϕ and ψ satisfy

$$\|\nabla\phi\|_{L^\infty(\mathbf{R}^{n-1})} \le M, \quad \text{and} \quad \|\nabla\psi\|_{L^\infty(\mathbf{R}^{n-2})} \le M.$$

We also assume that there exist $\delta_D \geq 1/M$ and $\delta_N \geq 1/M$ so that ϕ satisfies

$$\frac{\partial \phi}{\partial x_1} \geq \delta_N \text{ a.e. on } \{x' : x_1 > \psi(x'')\} \tag{3}$$

$$\frac{\partial \phi}{\partial x_1} \leq -\delta_D \text{ a.e. on } \{x' : x_1 < \psi(x'')\}. \tag{4}$$

These conditions are due to Brown [**3**], whose paper provides an example that illustrates their necessity. Let $\zeta = \{x \in \partial\Omega : x_1 = \psi(x'')\}$ be the boundary between N and D. We will refer to ζ as the "crease" of $\partial\Omega$ and we will call the graph domains we have described *creased domains*.

We will denote by ϕ_N the restriction of ϕ to $\{x' : x_1 \geq \psi(x'')\}$, and ϕ_D will denote the restriction of ϕ to $\{x' : x_1 < \psi(x'')\}$. In the construction below, we will need to have ϕ_D and ϕ_N defined on all of $\mathbf{R}^{n-1}$ so that the conditions (3) and (4) still hold. This is easily accomplished by setting

$$\begin{aligned} \phi_D(x_1, x'') &= \phi_D(\psi(x''), x'') - \delta_D(x_1 - \psi(x'')), \qquad x_1 \geq \psi(x'') \\ \phi_N(x_1, x'') &= \phi_D(\psi(x''), x'') + \delta_N(x_1 - \psi(x'')), \qquad x_1 \leq \psi(x''). \end{aligned}$$

In order to lighten the notation, we use the same notation for these extensions to $\mathbf{R}^{n-1}$. Note that with these definitions, we have $\phi = \max(\phi_D, \phi_N)$.

LEMMA 2.1. *For each x_1, the equation $x_1 = \phi_D(t, x'')$ has a unique solution and if we let $t = h(x_1, x'')$ denote the solution, then function h is Lipschitz on $\mathbf{R}^{n-1}$ and the Lipschitz constant depends only on M and δ_D.*

PROOF. Because of our assumption (4), the function $t \to \phi_D(t, x'')$ is strictly monotone and maps $\mathbf{R}$ onto $\mathbf{R}$. Hence, the existence of h follows. If we differentiate the equation

$$x_1 = \phi_D(h(x_1, x''), x'')$$

formally, we obtain

$$\begin{aligned} 1 &= \frac{\partial \phi_D}{\partial x_1}(h(x_1, x''), x'')\frac{\partial h}{\partial x_1}(x_1, x'') \\ 0 &= \frac{\partial \phi_D}{\partial x_1}(h(x_1, x''), x'')\frac{\partial h}{\partial x_j}(x_1, x'') + \frac{\partial \phi_D}{\partial x_j}(h(x_1, x''), x''), \quad j = 2, \ldots, n-2. \end{aligned}$$

Since we have $\partial\phi_D/\partial x_1 \leq -\delta_D$, we can solve for the gradient of h and obtain $\|\nabla h\|_{L^\infty(\mathbf{R}^{n-1})} \leq \max(M, 1)/\delta_D$. If ϕ_D were C^1 and not just Lipschitz, then the implicit function theorem would imply that h is differentiable and the formal calculation would imply that ∇h is bounded.

In the Lipschitz case, a little more work is needed. We regularize the function ϕ_D with a standard mollifier to obtain $\phi_{D,\varepsilon} = \phi_D * \eta_\varepsilon$. If we also assume that $\eta_\varepsilon \geq 0$, then we have that $\phi_{D,\varepsilon}$ also satisfies

$$\frac{\partial \phi_{D,\varepsilon}}{\partial x_1} \leq -\delta_D.$$

We let h_ε be the solution of $\phi_{D,\varepsilon}(h_\varepsilon(x_1, x''), x'') = x_1$. Since $\phi_{D,\varepsilon}$ is smooth, the argument above allows us to conclude that $\|\nabla h_\varepsilon\|_{L^\infty(\mathbf{R}^{n-1})}$ is bounded. We claim that h_ε converges to h uniformly as $\varepsilon \to 0^+$. To establish this claim, we observe that because ϕ_D is Lipschitz, then $\|\phi_{D,\varepsilon} - \phi_D\|_\infty \leq M\varepsilon$. Using this, the monotonicity of $\phi_{D,\varepsilon}$ and the definition of h, we obtain

$$\phi_{D,\varepsilon}(h(x_1,x'')+t,x'') \le x_1 + M\varepsilon - \delta_D t, \qquad t>0$$
$$\phi_{D,\varepsilon}(h(x_1,x'')+t,x'') \ge x_1 - M\varepsilon + \delta_D t, \qquad t<0.$$

Thus, we can conclude that $h_\varepsilon(x_1,x'')$ must satisfy

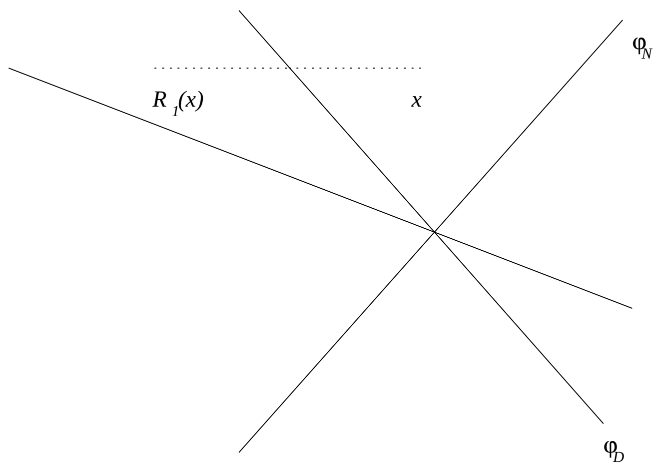
$$|h(x_1,x'') - h_\varepsilon(x_1,x'')| \le M\varepsilon/\delta_D.$$

Since, we have h_ε converges to h uniformly and $\|\nabla h_\varepsilon\|_{L^\infty(\mathbf{R}^{n-1})}$ is bounded independently of ε, it follows that h is Lipschitz. □

Using the function h, we define a reflection in the graph of ϕ_D by

$$R_1(x_1,x'',x_n) = (2h(x_n,x'') - x_1, x'', x_n).$$

As the figure indicates, this maps acts by mapping a point which is s units to



the right of the graph of ϕ_D to a point which is s units to the left of the graph of ϕ_D. It is clear that R_1 is Lipschitz and that $R_1 \circ R_1$ is the identity, hence R_1 is bi-Lipschitz. Next, we define $\Omega_1 = R_1(\Omega) \cup D \cup \Omega$. We claim that the image of the graph $\{(x',x_n) : x_n = \phi_N(x')\}$ is a Lipschitz graph on all of $\mathbf{R}^{n-1}$.

To see this, we would like to apply the implicit function theorem to write the set $\{(x',x_n) : x_n = \phi_N(R_1'(x',x_n))\}$ in the form $\{(x',x_n) : x_n = \gamma(x')\}$ (see figure above). As in Lemma 2.1, we cannot do this directly. Instead, we regularize, apply the implicit function theorem and take a limit. The details are omitted, see [**23**]. Now, we can write $\Omega_1 = \{(x',x_n) : x_n > \phi_1(x')\}$ where the function $\phi_1 = \max(\phi_N, \gamma)$.

Next, we define R_2 to be a reflection in $\partial\Omega_1$ by setting

$$R_2(x', \phi_1(x') + t) = (x', \phi_1(x') - t).$$

Using these reflections, we can change variables to obtain a divergence form operator $L = \operatorname{div} A\nabla$ with bounded measurable coefficients so that $\Delta u = 0$ in Ω if and only if $Lu \circ R_1 = 0$ in $R_1(\Omega)$ and so that $Lu = 0$ in Ω_1 if and only if $Lu \circ R_2 = 0$ in $R_2(\Omega)$. A similar construction is used by Dahlberg and Kenig, see [**8**] or [**23**] for details.

Finally, we introduce notation for function spaces. By $L^p(N)$, we mean the standard space of functions f with $|f|^p$ integrable with respect to surface measure on N. We denote by $L^{p,1}(D)$ the Sobolev space of functions with one derivative in $L^p(D)$. More precisely, $f \in L^{p,1}(D)$ if $f(x', \phi(x')) \in L^{p,1}(\{x' : x_1 < \psi(x'')\})$. We use the norm $\|f\|_{L^{p,1}(\partial\Omega)} = \|\nabla_{tan} f\|_{L^p(\partial\Omega)}$ on $L^{p,1}(D)$. Since we only consider $p \leq 2$ and $n \geq 3$, by Sobolev embedding on the $n-1$ dimensional boundary, we can choose a representative which vanishes at infinity, except when $n = 3$ and $p = 2$. In this case, we will only have uniqueness up to a constant.

In this section, we will prove an L^1 estimate for the nontangential maximal function of the gradient of the solution when the Neumann data is an atom and the Dirichlet data is zero. Recall that a bounded function a is called a $\partial\Omega$-atom if a is supported on a surface ball $B = B_r(y) \cap \partial\Omega = \{x \in \partial\Omega : |x - y| < r\}$ such that $\|a\|_{L^\infty(\partial\Omega)} \leq 1/\sigma(B)$ and $\int a\, d\sigma = 0$.

We call a function a an N-atom if $a = \bar{a}|_N$, where $\bar{a}$ is a $\partial\Omega$-atom. Note that an N-atom a may not have mean value 0, so a is not necessarily a $\partial\Omega$-atom. We then define the N-atomic Hardy space as the ℓ^1-span of the atoms a_j,

$$H^1_{at}(N) = \{f : f = \sum \lambda_j a_j \text{ with } \sum |\lambda_j| < \infty\}$$

with the norm of $f \in H^1_{at}(N)$ defined as

$$\|f\|_{H^1_{at}(N)} = \inf\{\sum |\lambda_j| : f = \sum \lambda_j a_j\}.$$

By this definition, $f \in H^1_{at}(N)$ if and only if $f = \bar{f}|_N$, where $\bar{f} \in H^1_{at}(\partial\Omega)$, and it also follows that

$$\inf\{\|\bar{f}\|_{H^1_{at}(\partial\Omega)} : f = \bar{f}|_N, \bar{f} \in H^1_{at}(\partial\Omega)\}$$

provides an equivalent norm on $H^1_{at}(N)$. See the papers of Chang, Krantz, and Stein [**6**] and Chang, Dafni, and Stein [**5**] for further discussion of defining Hardy spaces on domains in $\mathbf{R}^n$.

We consider the mixed problem with zero Dirichlet data and N-atomic Neumann data, a. Our goal is to show that for such a solution, we have the gradient in $L^1(\partial\Omega)$. Thus the main result of this section is the following proposition.

PROPOSITION 2.2. *Let Ω with N and D be a creased domain. If a is an N-atom and we solve the mixed problem (1) with data $f_D = 0$ and $f_N = a$, then the solution u satisfies*

$$\|N(\nabla u)\|_{L^1(\partial\Omega)} \leq C$$

where the constant C depends only on M.

In order to prove this estimate, we first need to know a solution exists. This is obtained from the earlier work of Brown [**3**].

PROPOSITION 2.3. *If Ω is a creased domain, $f_N \in L^2(N)$ and $\nabla_{tan} f \in L^2(D)$, then there is a unique solution to (1) with $N(\nabla u) \in L^2(\partial\Omega)$.*

PROOF. The existence is obtained by a straightforward extension of the arguments in Brown [**3**]. This paper considers the equation $\Delta - k^2$ for $k \in \mathbf{R} \setminus \{0\}$, the extension to $k = 0$ is straightforward. The uniqueness in graph domains may be proven using the idea of this paper. If we apply the divergence theorem, we obtain that

$$\int_{\partial\Omega} |\nabla u|^2 e_1 \cdot \nu - 2\frac{\partial u}{\partial \nu} e_1 \cdot \nabla u \, d\sigma = 0$$

since div $(|\nabla u|^2 e_1 - 2\nabla u e_1 \cdot \nabla u) = 0$. This is a version of the Rellich identity, see [**18, 13**]. The condition $N(\nabla u) \in L^2(\partial\Omega)$ is used to justify the integration by parts.

The hypotheses on Ω are designed so that the vector e_1 satisfies $e_1 \cdot \nu > \delta_N/\sqrt{1+M^2}$ a.e. on N and $e_1 \cdot \nu \leq -\delta_D/\sqrt{1+M^2}$ a.e. on D. Rearranging terms and using Cauchy's inequality gives

$$(5) \qquad \int_N |\nabla_{tan} u|^2 \, d\sigma + \int_D \left|\frac{\partial u}{\partial \nu}\right|^2 \, d\sigma \leq C \left(\int_D |\nabla_{tan} u|^2 \, d\sigma + \int_N \left|\frac{\partial u}{\partial \nu}\right|^2 \, d\sigma \right).$$

The estimate (5) tells us that that if u has zero data in the mixed problem, then $\nabla u = 0$ on $\partial\Omega$. Now uniqueness in the regularity problem tells us that $u = 0$. See the work of Dahlberg and Kenig [**8**] for uniqueness of the regularity problem in graph domains. □

A key step in the proof of Proposition 2.2 is to consider solutions of a divergence form operator in the complement of a ball. There is an asymptotic expansion for these solutions which allows us to conclude that the solution has some extra decay if a certain integral vanishes. This is an old result of Serrin and Weinberger [**20**] which recall here. If u is a solution of a elliptic divergence form operator div $A\nabla$ in the complement of a ball, $\mathbf{R}^n \setminus B_1(x_0)$, then we have the following asymptotic expansion for u

$$(6) \qquad u(x) = u_\infty + \beta G(x) + w(x)$$

where u_∞ is a constant. The function G is the green's function with pole at x_0 which is known to satisfy $G(x) \leq C|x - x_0|^{2-n}$. The constant β is the outflow which is given by

$$\beta = \int_{\mathbf{R}^n} A(x)\nabla u(x) \cdot \nabla\psi(x) \, dx$$

where ψ is any Lipschitz function which is one in a neighborhood of ∞ and zero in a neighborhood of $\{x : |x - x_0| \leq 1\}$. Finally, v is a function which satisfies

$$|v(x)| \leq C|x - x_0|^{2-n-\delta} \int_{2<|x-x_0|<3} |u(x)| \, dx$$

for constant C and $\delta > 0$ which depend only on the dimension and the bounds on the eigenvalues of the matrix of coefficients A.

Proof. Now we turn to the proof of Proposition 2.2. We consider an N-atom a. Since the estimate we desire to prove and the class of creased domains are invariant under dilation, we may assume that the ball appearing the definition of an N-atom has radius $r = 1$. According to Proposition 2.3, there is a solution u to (1) with $f_D = 0$ and $f_N = a$ and with

$$\|N(\nabla u)\|_{L^2(\partial\Omega)} < \infty.$$

Our goal is to show that ∇u decays sufficiently fast at infinity so that $N(\nabla u) \in L^1(\partial\Omega)$. We extend u by odd reflection about D using the map R_1 defined above to obtain u in Ω_1, satisfying

$$\begin{aligned} u_1 \circ R_1 &= -u_1 \\ Lu_1 &= 0 \end{aligned}$$

for the divergence form elliptic operator L we defined above. Since $N(\nabla u) \in L^2(\partial\Omega)$, it follows that $\nabla u_1 \in L^2(B)$ for any bounded set $B \subset \Omega_1$ and thus that

u_1 is a weak solution of the Neumann problem

$$\begin{cases} Lu = 0 & \text{in } \Omega_1 \\ A\nabla u \cdot \nu = a + a^* & \text{on } \partial\Omega_1 \end{cases}$$

where $a^*(y) = -(DR_1 \circ R_1^{-1})^t \nu(y) \cdot \nu(R_1^{-1}(y))a(R_1(y))$. Since we do not yet know that ∇u is in $L^2(\Omega)$, we require that the test functions in the weak formulation of this problem be zero outside some large ball. However, we do not require test functions to vanish on $\partial\Omega$. Changing variables, we see that

$$\int_N a(y)\, d\sigma(y) = -\int_{R_1(N)} a^*(y)\, d\sigma(y).$$

Thus, if $\operatorname{dist}(\operatorname{supp} a, D) < 1$, then $\lambda(a + a^*)$ is an atom for $\partial\Omega_1$, where the constant λ depends only on M. If $\operatorname{dist}(\operatorname{supp} a, D) > 1$, then we have that a is a $\partial\Omega_1$ atom and λa^* is a multiple of a $\partial\Omega_1$ atom for λ depending only on permissible quantities.

We may reflect in $\partial\Omega_1$ using R_2 to obtain an even extension u_2 which solves $Lu_2 = 0$ in $\mathbf{R}^n \setminus \operatorname{supp}(a + a^*)$. Applying the theorem of Serrin and Weinberger (6),

$$u_2(x) = \beta g(x) + w(x) \tag{7}$$

where $|w(x)| = O(|x|^{2-n-\delta})$ for some $\delta > 0$. We have that $u_\infty = 0$ since u vanishes on D. Now the estimate $|u_2(x)| \le C|x|^{2-n}$ for x large and Caccioppoli's inequality implies $\nabla u \in L^2(\mathbf{R}^n)$, $n \ge 3$. Thus we can conclude that the solution u_2 satisfies

$$\|\nabla u_2\|_{L^2(\mathbf{R}^n)} < \infty. \tag{8}$$

Our next step is to claim that there exist $C > 0$ and $\delta > 0$ so that

$$|u(x)| \le C|x|^{2-n-\delta}, \qquad x \in \Omega. \tag{9}$$

In this estimate, the constant C depends only on the constant M. In order to obtain the correct dependence of the constant, we must be more careful than we were in the previous paragraph. This requires consideration of the cases $\operatorname{dist}(\operatorname{supp} a, D) \le 1$ and $\operatorname{dist}(\operatorname{supp} a, D) > 1$ separately.

We first consider $\operatorname{dist}(\operatorname{supp} a, D) \le 1$. We recall that in the expansion (7), $C_0 = 0$ since $u = 0$ on D and

$$\beta = \int_{\mathbf{R}^n} A(x)\nabla\eta(x) \cdot \nabla u_2(x)\, dx$$

where η is any function which is zero on the support of the Neumann data a and is identically one for all x sufficiently large.

As in Dahlberg and Kenig [**8**, p. 444], we have

$$\int_{\mathbf{R}^n} A(x)\nabla\eta(x)\nabla u(x)\, dx = 2\int_{\partial\Omega_1} (a(x) + a^*(x))\, d\sigma(x) = 0$$

and hence $\beta = 0$. Now, we can estimate u in the set of points which are at most distance 2 from the support of a using the $L^2(\partial\Omega)$ estimate for $N(\nabla u)$. Since u vanishes on D, a Poincaré inequality gives us an estimate for the integral of $|u|^2$ in Ω near support a. This gives the estimate (9) with correct dependence of the constant.

If $\operatorname{dist}(\operatorname{supp} a, D) > 1$, then the above procedure gives the correct order at infinity, but the constant C depends on the distance between the support of a and

a^*. In this case, we need to split the solution $u_1 = u_{11} + u_{12}$ where u_{11} is a weak solution of the Neumann problem

$$\begin{cases} Lu_{11} = 0, & \text{in } \Omega_1 \\ \frac{\partial u_{11}}{\partial \nu} = a, & \text{on } \partial\Omega_1. \end{cases}$$

The function u_{12} solves the same problem with data a^* instead of a. We claim these solutions each satisfy the estimates

$$(10)\quad |u_{11}(x)| \leq C\,\mathrm{dist}\,(x, \mathrm{supp}\ a)^{2-n-\delta}, \qquad x \in \Omega_1,\ \mathrm{dist}\,(x, \mathrm{supp}\ a) > 1$$

$$(11)\quad |u_{12}(x)| \leq C\,\mathrm{dist}\,(x, \mathrm{supp}\ a^*)^{2-n-\delta}, \qquad x \in \Omega_1,\ \mathrm{dist}\,(x, \mathrm{supp}\ a^*) > 1$$

Together, (10) and (11) imply the estimate (9) in the second case.

The proof of (10) and (11) depends on reflecting in Ω_1 and then using the Serrin-Weinberger theorem (6). Again, we have $u_\infty = 0$ since the energy solutions are in $L^{2n/(2n-2)}(\mathbf{R}^n)$ and $\beta = 0$ since a and a^* have mean value zero. The dependence of the constant (and the existence of u_{11} and u_{12}) depends on the following boundary Poincaré inequality. In this Lemma, we use $⨍_E$ to denote the average over E.

Lemma 2.4. *If Ω is a Lipschitz graph domain, u is a function with ∇u in $L^2(\Omega)$, $x \in \partial\Omega$ and $r > 0$, then with*

$$\bar{u} = ⨍_{B_r(x)\cap\partial\Omega} u(x)\, d\sigma$$

we have that

$$\int_{\partial\Omega\cap B_r(x)} |u(x) - \bar{u}|^2\, d\sigma(x) \leq Cr^2 \int_{\Omega\cap B_r(x)} |\nabla u(x)|^2\, dx.$$

As a consequence of this Lemma, we can show that the map

$$u \to \int_{\partial\Omega_1} au\, d\sigma$$

is a continuous linear functional on the Sobolev space $\dot{L}^{2,1}(\Omega_1)$ which we norm with $(\int_{\Omega_1} |\nabla u|^2\, dx)^{1/2}$. Thus, we can use Lax-Milgram to show that there is a weak solution of the Neumann problem for L with data a and that this solution satisfies $\|\nabla u\|_{L^2(\Omega_1)} \leq C$ (the constant depends on r in general, but here we have no r dependence since we have already set $r = 1$). Then, Sobolev embedding implies that the solution lies in $L^{2n/(2n-2)}(\Omega_1)$ and this can be used to show that the constants in (10) and (11) depend only on the Lipschitz constant M. We also note that the estimate (8) and uniqueness in the class of solutions with ∇u in $L^2(\Omega_1)$ is needed to show $u_1 = u_{11} + u_{12}$.

Now from the estimate (9) we can use Caccioppoli's inequality to find that on the ring

$$R_k = \{x : 2^k \leq \mathrm{dist}\,(x, \mathrm{supp}\ a) < 2^{k+1}\},$$

we have

$$(12)\qquad \int_{R_k\cap\Omega} |\nabla u|^2\, dx \leq C2^{k(2-n-2\delta)}, \qquad k = 0, 1, \dots$$

for some constant $C = C(M)$ and for δ as in the Serrin-Weinberger theorem (see (9)).

Our next step is to obtain estimates for ∇u on the boundary. We let $\eta \geq 0$ be a cutoff function with $\eta = 1$ on R_k and $\mathrm{supp}\ \eta \subset \cup_{|j|\leq 1} R_{k+j}$ and $|\nabla\eta| \leq C2^{-k}$.

We again use the Rellich identity as in the proof of Proposition 2.3, but with e_1 replaced by ηe_1 to obtain that

$$\begin{aligned}\int_{R_k\cap\partial\Omega}|\nabla u(x)|^2\,d\sigma(x) &\leq C\int_{\partial\Omega}\eta(x)|\nabla u(x)|^2\,d\sigma(x)\\ &\leq C\int_{\Omega}|\nabla\eta(x)||\nabla u(x)|^2\,dx.\end{aligned}$$

Now, Caccioppoli's inequality (12) and our observation that u has mixed data 0 away from supp a gives that

$$\int_{R_k\cap\partial\Omega}|\nabla u(x)|^2\,dx \leq C2^{k(1-n-2\delta)},\qquad k\geq 1.$$

Finally, the localization argument as in Dahlberg and Kenig [**8**, p. 445] gives

$$\int_{R_k}N(\nabla u)(x)^2\,d\sigma(x)\leq C2^{k(1-n-2\delta)}.$$

This estimate implies that we have

$$\int_{\partial\Omega}N(\nabla u)(x)\,d\sigma(x)\leq C.$$

This concludes the proof of the existence of solutions as described in Proposition 2.2. □

Finally, we give the main result of this section. The existence of solutions with atomic data. Before we can do this, we must define the space $H^{1,1}_{at}(D)$, the space of functions with one derivative in the Hardy space. The simplest way to define this is to observe that, like $H^1(N)$, $H^{1,1}_{at}(D)$ is the restriction to D of the corresponding space on $\partial\Omega$. Thus, we say b is a 1-atom for $\partial\Omega$ if b is supported in a ball $B_r(x)\cap\partial\Omega$ and $\|\nabla_{tan}b\|_{L^\infty(B_r(x)\cap\partial\Omega)} < r^{2-n}$. As before, b is 1-atom for D if $b=\bar{b}|_D$ for $\bar{b}$ a 1-atom for $\partial\Omega$. Finally, the space $H^{1,1}_{at}(D)$ is defined as the ℓ^1-span of 1-atoms for D.

Theorem 2.5. *Let Ω, N and D give a creased domain. If f_N is in $H^1(N)$ and f_D is in $H^{1,1}_{at}(D)$, then there exists a solution of the mixed problem (1) which satisfies*

$$\|N(\nabla u)\|_{L^1(\partial\Omega)}\leq C(\|f_N\|_{H^1_{at}(N)}+\|f_D\|_{H^{1,1}_{at}(D)}).$$

Furthermore, if the f_N is also in $L^2(N)$ and f_D is also in $L^{2,1}(D)$, then the solution satisfies $N(\nabla u)\in L^1(\partial\Omega)\cap L^2(\partial\Omega)$.

Proof. The result for Hardy spaces follows from Proposition 2.2. To see this, we first use the result of Dahlberg and Kenig on the regularity of the solution of the Dirichlet problem when the Dirichlet data is in space $H^{1,1}_{at}(\partial\Omega)$. This result allows us to reduce the case when f_D is zero. For this special case, we can write the data f_N as a sum of atoms and then add the solutions from Proposition 2.2.

The additional statements about L^2, follow from the observation of Pipher and Verchota [**17**] that if we take a $L^2\cap H^1$ function, then the atomic decomposition can be chosen so that the sum converges in L^2 also. This technical observation is needed in the interpolation argument in section 4. □

3. Uniqueness

Now we turn to uniqueness. This seems to be rather involved. The proof for uniqueness in the L^p problem is the same as for the Hardy space problem, thus we do them simultaneously.

PROPOSITION 3.1. *If Ω is a creased domain, $N(\nabla u) \in L^p(\partial\Omega)$, $1 \leq p < 2$, $n \geq 3$ and u is a solution of the mixed problem with $f_D = 0$ and $f_N = 0$, then $u = 0$.*

PROOF. The argument proceeds by duality. Thus, we consider a solution w of the mixed problem with Dirichlet data g_D, a Lipschitz function with compact support in $\mathbf{R}^n$ and zero Neumann data. Thus w satisfies:

$$\begin{cases} \Delta w = 0, & \text{in } \Omega \\ w = g_D, & \text{on } D \\ \frac{\partial w}{\partial \nu} = 0, & \text{on } N \end{cases}$$

The function g_D will be a multiple of a 1-atom on D for the space $H^{1,1}_{at}(D)$. A straightforward adaptation of the argument for existence of solutions with atomic Neumann data allows us to show that the solution w satisfies

$$N(\nabla w) \in L^1(\partial\Omega) \cap L^2(\partial\Omega). \tag{13}$$

Next, we observe that the solution w lies in the Hölder space $C^\alpha(\bar{\Omega})$ for some $\alpha = \alpha(M) > 0$ and thus

$$\|u\|_{C^\alpha(\bar{\Omega})} = \sup_{x \neq y, x, y \in \Omega} \frac{|w(x) - w(y)|}{|x - y|^\alpha} \leq C(g_D). \tag{14}$$

The equality serves to define the Hölder semi-norm. The inequality in (14) is probably well-known. It can be proven by reflecting w in the Neumann face N and using the Hölder continuity at the boundary of weak solutions of the Dirichlet problem (see [**11**]).

Since we have $N(\nabla w)$ is in $L^1(\partial\Omega) \cap L^2(\partial\Omega)$, we can use a version of the Sobolev embedding theorem to conclude that

$$N(w) \in L^{(n-1)/(n-2)}(\partial\Omega) \tag{15}$$

(see Brown [**4**] for a proof).

Next, we recall the (generalized) Riesz transforms on a Lipschitz graph domain. If $N(w) \in L^p(\partial\Omega)$, $p < \infty$, then with $r = \delta(x)/2$, we have

$$|\nabla w(x)| \leq \frac{C}{r} \fint_{B_r(x)} |w(x)|\, dx \leq C\delta(x)^{-1-(n-1)/p} \|N(w)\|_{L^p(\partial\Omega)}.$$

Thus, we define for $j = 1, \ldots, n$

$$w_j(x) = -\int_{x_n}^\infty \frac{\partial w}{\partial x_j}(x', t)\, dt. \tag{16}$$

The collection $w_1, \ldots, w_n$ satisfy

$$\begin{aligned} w_n &= w \\ \frac{\partial w_j}{\partial x_k} &= \frac{\partial w_k}{\partial x_j} \qquad j, k = 1, \ldots, n \\ \sum_{j=1}^n \frac{\partial w_j}{\partial x_j} &= 0. \end{aligned} \tag{17}$$

The last two equations in (17) are the system for conjugate harmonic functions as studied by Stein and Weiss [**22**]. Using square function arguments as in Stein's book, [**21**], one can show that

$$\|N(w_i)\|_{L^p(\partial\Omega)} \le C\|N(w)\|_{L^p(\partial\Omega)}, \qquad 0 < p < \infty. \tag{18}$$

The argument for the half-space as presented by Stein extends immediately to a Lipschitz graph domain, once we have the equivalence of the non-tangential maximal function and the square function as proven by Dahlberg [**7**].

Next, we rely on additional ideas from Littlewood-Paley theory to show that each w_i is also in $C^\alpha(\bar\Omega)$. This is proven using two observations about harmonic functions in Lipschitz graph domains. First, if w is C^α, $0 < \alpha \le 1$, and harmonic, then for each multi-index β with $|\beta| > 0$,

$$\sup_{x\in\Omega} \delta(x)^{|\beta|-\alpha} \left|\frac{\partial^\beta w}{\partial x^\beta}(x)\right| \le C\|w\|_{C^\alpha(\bar\Omega)}. \tag{19}$$

This is proven by estimating the derivative of $w(\cdot)$ by the average of $w(\cdot) - w(x)$ on a ball of radius half the distance to the boundary. Conversely, if w is differentiable in a Lipschitz graph domain Ω, then

$$\|w\|_{C^\alpha(\bar\Omega)} \le \sup_{x\in\Omega} \delta(x)^{1-\alpha}|\nabla w(x)|. \tag{20}$$

From (14), we have $w = w_n$ is Hölder continuous in Ω and thus the second derivatives of w satisfy the condition (19). If we differentiate the definition of w_i, (16), and use (19) we obtain that the gradient of each w_i satisfies

$$\sup \delta(x)^{1-\alpha}|\nabla w_i(x)| \le C\|w_n\|_{C^\alpha(\bar\Omega)}.$$

Hence, from (20) we have that

$$\|w_i\|_{C^\alpha(\bar\Omega)} \le C(g_D). \tag{21}$$

Next, we observe that if $\|N(w)\|_{L^p(\Omega)} < \infty$ and $\|w\|_{C^\alpha(\bar\Omega)} < \infty$, then w must be bounded. To see this, use the nontangential maximal function to obtain that w is bounded when $\delta(x) > 1$ and then the Hölder continuity near the boundary. Applying this observation to each w_i, we conclude that each of the w_i is bounded in Ω. Combining the estimate (21) with the estimates (15) and (18) for each w_i, we obtain

$$\|N(w_i)\|_{L^{(n-1)/(n-2)}(\partial\Omega)} + \|w_i\|_{L^\infty(\bar\Omega)} \le C(g_D). \tag{22}$$

Now, we have u as in Proposition 3.1, a solution of the mixed problem with $N(\nabla u) \in L^p(\partial\Omega)$ and zero data in the mixed problem (1). For $s > 0$ we let u_s denote a translation of u defined by $u_s(x) = u(x + se_n)$ and define $w_t(x) = w(x + te_n)$. We consider u_s, w_t and a cutoff function η_R which is one on a ball of radius R and 0 outside a ball of radius $2R$. Applying Green's second identity gives

$$\int_{\partial\Omega} \eta_R\Big(u_t\frac{\partial w_s}{\partial\nu} - w_s\frac{\partial u_t}{\partial\nu}\Big)\, d\sigma = \int_\Omega \nabla\eta_R(x)\cdot(u_t(x)\nabla w_s(x) - w_s(x)\nabla u_t(x))\, dx.$$

We consider, in turn, the various limits as t and s approach zero and R approaches infinity. First, consider R going to infinity. Since $N(\nabla u)$ is in $L^p(\partial\Omega)$, the Sobolev embedding Lemma in [**4**]

$$N(u_t) \in L^{p(n-1)/(n-1-p)}(\partial\Omega), \qquad t \ge 0. \tag{23}$$

Also, we have $N(\nabla w_s) \in L^1(\partial\Omega)$ and for $s > 0$, $|\nabla w_s| \le C_s$, when $s > 0$. Thus,

$$\int_\Omega |\nabla \eta_R \cdot (u_t \nabla w_s)|\, dx \le C \int_{\{x:|x|\ge cR\}\cap\partial\Omega} N(u_t) N(\nabla w_s)\, d\sigma$$

which vanishes in the limit as $R \to \infty$ since u satisfies (23) and we have $N(\nabla w_s)$ is in $L^1(\partial\Omega) \cap L^\infty(\partial\Omega)$ (though with a constant which depends on s). Next, we consider

$$\int_\Omega |w_s \nabla \eta_R \cdot \nabla u_t|\, dx \le C \int_{\{x:|x|>cR\}} N(w_s) N(\nabla u_t)\, d\sigma.$$

Since we have $N(\nabla u_t) \in L^p(\partial\Omega)$ for some p in $[1, 2)$, and $w_s \in L^{(n-1)/(n-2)}(\partial\Omega) \cap L^\infty$, we have that this term also vanishes in the limit as $R \to \infty$. Finally, we may use dominated convergence to find the limit

$$\lim_{R\to\infty} \int_{\partial\Omega} \eta_R (u_t \frac{\partial w_s}{\partial \nu} - w_s \frac{\partial u_t}{\partial \nu})\, d\sigma = 0.$$

This is because (as we have shown above) the expression multiplying η_R is in $L^1(\partial\Omega)$. Thus, we have shown for all $s > 0$ and all $t > 0$ that

$$\int_{\partial\Omega} u_t \frac{\partial w_s}{\partial \nu} - w_s \frac{\partial u_t}{\partial \nu}\, d\sigma = 0. \tag{24}$$

Our next step is to let s and then t go to zero. We begin by showing that

$$\lim_{s\to 0^+} \int_{\partial\Omega} u_t \frac{\partial w_s}{\partial \nu}\, d\sigma = \int_{\partial\Omega} u_t \frac{\partial w}{\partial \nu}\, d\sigma. \tag{25}$$

We take advantage of the fact that for each $t > 0$, u_t is smooth and hence bounded in Ω. The function $N(\nabla w)$ is in $L^1(\partial\Omega)$ and hence (25) follows easily from the dominated convergence theorem. Next, we consider

$$\lim_{t\to 0^+} \int_{\partial\Omega} w_s \frac{\partial u_t}{\partial \nu}\, d\sigma = \int_{\partial\Omega} w_s \frac{\partial u}{\partial \nu}\, d\sigma. \tag{26}$$

This follows since we have $\nabla u \in L^p(\partial\Omega)$ for some p, $1 \le p < 2$ and w satisfies (22). The same estimates allow us to use the dominated convergence theorem to show

$$\lim_{s\to 0^+} \int_{\partial\Omega} w_s \frac{\partial u}{\partial \nu}\, d\sigma = \int_D g_D \frac{\partial u}{\partial \nu}\, d\sigma. \tag{27}$$

The integral on the right-hand side of this equation is only over D since $\frac{\partial u}{\partial \nu} = 0$ on N.

After these routine limits, we now come to the interesting part of the argument. We claim that

$$\lim_{s\to 0^+} \int_{\partial\Omega} u \frac{\partial w_s}{\partial \nu}\, d\sigma = 0. \tag{28}$$

This presents a challenge since (if say, $p = 1$) we have estimates for u in $L^{(n-1)/(n-2)}(\partial\Omega)$, but we only have $N(\nabla w)$ in $L^p(\partial\Omega)$ for $p \le 2$.

Our argument relies on a technique from G. Verchota's paper [**25**] which uses the generalized Riesz transforms to express the integrand in (28) in terms of derivatives of u and the boundary values of w_i. These quantities are easily estimated as in our first three limits.

To proceed, we write out the integral over $\partial\Omega$ in the coordinates $(x', \phi(x'))$ and the use the system that $w_1, \ldots, w_n$ satisfies to obtain

$$\begin{aligned}
(29)\qquad \int_{\partial\Omega} u \frac{\partial w_s}{\partial \nu}\, d\sigma &= \int_{\mathbf{R}^{n-1}} u(x', \phi(x'))(\sum_{i=1}^{n-1} \phi_{x_i}(x') \frac{\partial w_s}{\partial x_i}(x', \phi(x')) \\
&\qquad - \frac{\partial w_s}{\partial x_n}(x', \phi(x')))\, dx' \\
(30)\qquad &= \int_{\mathbf{R}^{n-1}} u(x', \phi(x'))(\sum_{i=1}^{n-1} \phi_{x_i}(x') \frac{\partial w_{i,s}}{\partial x_n}(x', \phi(x')) \\
(31)\qquad &\qquad + \frac{\partial w_{i,s}}{\partial x_i}(x', \phi(x')))\, dx' \\
&= -\int_{\mathbf{R}^{n-1}} \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} u(x', \phi(x')) w_{i,s}(x', \phi(x'))\, dx'.
\end{aligned}$$

The boundary terms at infinity in the integration by parts vanish since we have $w_i \in L^{(n-1)/(n-2)}(\partial\Omega) \cap L^\infty(\partial\Omega)$ and $u \in L^{p(n-1)/(n-1-p)}(\partial\Omega)$. We omit the details. Now, as above, we may use $w_i \in L^q(\partial\Omega)$, for all q with $(n-1)/(n-2) \leq q \leq \infty$ and that $\nabla u \in L^p(\partial\Omega)$, for some p, $1 \leq p < 2$ to conclude that

$$\lim_{s \to 0^+} \int_{\partial\Omega} w_{i,s}(x', \phi(x')) \frac{\partial}{\partial x_i} u(x', \phi(x'))\, dx' = \int_{\mathbf{R}} w_i(x', \phi(x')) \frac{\partial}{\partial x_i} u(x', \phi(x'))\, dx'.$$

Now we have the task of reversing the calculations in (31) to conclude that this last integral is zero.

To do this, we first observe that, away from the crease, we have some additional regularity of the solution u. This follows from the solution of the Neumann and regularity problem as in Dahlberg and Kenig, [**8**]. In particular, using the Green's function for each of these problems as in [**2**, p. 33] (which is based on [**8**]), we can show that if B_r is a ball centered at a point in $\partial\Omega$ and $B_{2r} \cap \partial\Omega$ is contained in N or in D, but does not meet both sets, then we have

$$\sup_{y \in B_r(x) \cap \Omega} |u(y)| \leq C \int_{B_{2r}(x) \cap \partial\Omega} N(u)\, d\sigma. \qquad (32)$$

Let η_ε be a cutoff function which is one if $\operatorname{dist}(x, D) > 2\varepsilon$, is zero when $\operatorname{dist}(x, D) < \varepsilon$ and satisfies $|\nabla \eta_\varepsilon| \leq C/\varepsilon$. Also, let $\tilde{\eta}_R$ be a cutoff function which is one if $|x| < R$, zero if $|x| > 2R$ and satisfies $|\nabla \tilde{\eta}_R| \leq C/R$. Then since the integrand is in $L^1(\mathbf{R}^{n-1})$,

$$\begin{aligned}
&\int_{\mathbf{R}^{n-1}} w_i(x', \phi(x')) \frac{\partial}{\partial x_i} u(x', \phi(x'))\, dx' \\
&= \lim_{R \to \infty, \varepsilon \to 0^+} \int_{\mathbf{R}^{n-1}} \eta_\varepsilon(x', \phi(x')) \tilde{\eta}_R(x', \phi(x')) w_i(x', \phi(x')) \frac{\partial}{\partial x_i} u(x', \phi(x'))\, dx'
\end{aligned}$$

since $\nabla u \in L^p$ for some p, $1 \leq p < 2$ and w_i satisfies (22). Note that we can assume the cutoff functions vanish on D since the tangential derivatives of u are already zero there. Now we consider the integral inside the limit with $\varepsilon > 0$ and integrate by parts to obtain

$$\begin{aligned}
&\int_{\mathbf{R}^{n-1}} \tilde\eta_R(x',\phi(x'))\eta_\varepsilon(x',\phi(x'))\, w_i(x',\phi(x'))\frac{\partial}{\partial x_i}u(x',\phi(x'))\,dx' \\
&\quad = \quad -\int_{\mathbf{R}^{n-1}} (\tilde\eta_R(x',\phi(x'))\eta_\varepsilon(x',\phi(x'))u(x',\phi(x'))\frac{\partial}{\partial x_i}w_i(x',\phi(x')) \\
&\qquad\qquad +u(x',\phi(x'))w_i(x',\phi(x'))\frac{\partial}{\partial x_i}(\tilde\eta_R(x',\phi(x'))\eta_\varepsilon(x',\phi(x')))\,dx'
\end{aligned}$$

where the boundary term at infinity vanishes because of our estimates on w_i and u. Since the estimate (32) implies u is bounded away from the crease and $N(\nabla w_i)$ is in $L^1(\partial\Omega)$, we may undo the calculation in (31) above to show that

$$\begin{aligned}
&\sum_{i=1}^{n-1}\int_{\mathbf{R}^{n-1}} \frac{\partial}{\partial x_i}w_i(x',\phi(x'))u(x',\phi(x'))\tilde\eta_R(x',\phi(x'))\eta_\varepsilon(x',\phi(x'))\,d\sigma \\
&\quad = \quad \int_{\partial\Omega} w\frac{\partial u}{\partial\nu}\tilde\eta_R\eta_\varepsilon\,d\sigma = 0.
\end{aligned}$$

Of course, this is because $\partial w/\partial\nu = 0$ on N and $u=0$ on D. Thus, to finish, we need to show that

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_N |w_i u\frac{\partial}{\partial x_i}\eta_\varepsilon|\,d\sigma &= 0\\
\lim_{R\to\infty}\int_N |w_i u\frac{\partial}{\partial x_i}\tilde\eta_R|\,d\sigma &= 0
\end{aligned}$$

We consider the limit involving ε first. Here, we use that $u=0$ on D to prove a Poincarè inequality: For some $C>0$, we have

$$\int_{\{x:\mathrm{dist}\,(x,\zeta)\le 2\varepsilon\}} |u(x)|^p\,d\sigma(x) \le C\varepsilon^p\int_{\{x\in\partial\Omega:\mathrm{dist}\,(x,\zeta)\le C\varepsilon\}}|\nabla u(x)|^p\,d\sigma.$$

Finally, if we let ζ_ε denote $\{x:\mathrm{dist}\,(x,\zeta)<C\varepsilon\}$ then we have

$$\int_N |uw_i\frac{\partial}{\partial x_i}\eta_\varepsilon|\,d\sigma \le \lim_{\varepsilon\to 0^+} C\left(\int_{\zeta_\varepsilon}|\nabla u(x)|^p\,d\sigma\right)^{1/p}\left(\int_{\zeta_\varepsilon}|w_i(x)|^{p'}\right)^{1/p'} = 0.$$

This uses that ∇u is in $L^p(\partial\Omega)$ and w_i is in $L^{p'}(\partial\Omega)$. The limit that $R\to\infty$ vanishes because we have w_i in $L^{(n-1)/(n-2)}(\partial\Omega)$ and $u\in L^{p(n-1)/(n-1-p)}(\partial\Omega)$. This completes the proof of the claim (28).

Combining (24), (25), (26), (27) and (28), we have that $\partial u/\partial\nu$ is zero on $\partial\Omega$. Thus the uniqueness for the Neumann problem proved in [**8**] implies that u is zero. This completes the proof of uniqueness for p in the range $1\le p<2$. □

4. Interpolation

Our last step, is to show existence of solutions for p in the range, $1<p<2$. This is done by interpolation. Only one or two details are needed.

PROPOSITION 4.1. *Let Ω, N and D be a creased domain. If f_D is in $L^{p,1}(D)$ and $f_N\in L^p(N)$, then there exists a solution u to (1) which satisfies*

$$\|N(\nabla u)\|_{L^p(\partial\Omega)} \le C(\|f_N\|_{L^p(\partial\Omega)} + \|f_D\|_{L^{p,1}(\partial\Omega)}).$$

Proof. Thanks to the solvability of the Dirichlet problem with data f which has $\nabla_{tan} f \in L^p(\partial\Omega)$ [**8**, Theorem 3.8], we may assume that $f_D = 0$. If f is in $L^p(N)$, $1 < p < 2$, then we may extend f to all of $\partial\Omega$ by setting $f = 0$ on D. We consider the function

$$\tilde{f}(x') = f(x', \phi(x'))\sqrt{1 + |\nabla\phi(x')|^2}.$$

According to the decomposition lemma in the monograph of Torchinsky, [**24**, p. 364], we have for $\lambda > 0$ that we can write

$$\tilde{f} = f_\lambda + f^\lambda$$

where

$$\|f^\lambda\|_{H^1(\mathbf{R}^{n-1})} \le C\lambda^{1-p}\|f\|_p^p \quad \text{and} \quad \|f_\lambda\|_{L^2(\mathbf{R}^{n-1})} \le C\lambda^{2-p}.$$

Thus we may use uniqueness in the L^2-mixed problem and the observation about L^2-solutions in Theorem 2.5 to show that

$$u = u^\lambda + u_\lambda$$

where u is the solution of (1) with data $f_N = f|_N$ and $f_D = 0$ and u_λ has Neumann data $f_N(x', \phi(x')) = (1 + |\nabla\phi(x')|^2)^{-1/2} f_\lambda(x')$ and $f_D = 0$. The function u^λ solves a mixed problem with data involving f^λ.

Next, from our estimates for H^1 and L^2, we obtain the weak-type estimate,

$$\sigma(\{x : N(\nabla u) > \lambda\}) \le \sigma(\{N(\nabla u^\lambda) > \lambda/2\}) + \sigma(\{N(\nabla u_\lambda) > \lambda/2\}) \le \frac{C}{\lambda^p}\|f\|_p^p.$$

This weak type estimate and the Marcinkiewicz interpolation theorem (see [**21**, p. 272]) give the strong-type inequality. Thus we have that if $f \in L^2 \cap L^p$, there is a solution to (1) which satisfies

$$\|N(\nabla u)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}.$$

To remove the restriction that f is in $L^2 \cap L^p$, involves a routine limiting argument. $\square$

Finally, we summarize what we have done. The result of this theorem is the main result of this paper.

Theorem 4.2. *Suppose $1 < p < 2$ and $n \ge 3$. Let Ω, N and D be a creased domain.*

If f_N is in $L^p(N)$ and f_D is in $L^{p,1}(D)$, then there exists a unique solution to the mixed problem (1) which satisfies

$$\|N(\nabla u)\|_{L^p(\partial\Omega)} \le C(\|f_N\|_{L^p(N)} + \|f_D\|_{L^{p,1}(D)}).$$

If f_N is in $H^1_{at}(N)$ and f_D is in $H^{1,1}_{at}(D)$, then there exists a unique solution to the mixed problem (1) which satisfies

$$\|N(\nabla u)\|_{L^1(\partial\Omega)} \le C(\|f_N\|_{H^1_{at}(N)} + \|f_D\|_{H^{1,1}_{at}(D)}).$$

In each case, the uniqueness assertion is in the class of solutions which have $N(\nabla u)$ in $L^p(\partial\Omega)$. When $p = 2$ and $n = 3$, we only have uniqueness modulo constants.

References

[1] A. Azzam and E. Kreyszig. On solutions of elliptic equations satisfying mixed boundary conditions. *SIAM J. Math. Anal.*, 13(2):254–262, 1982.

[2] R. M. Brown. The initial-Neumann problem for the heat equation in Lipschitz cylinders. *Trans. Amer. Math. Soc.*, 320:1–52, 1990.

[3] R. M. Brown. The mixed problem for Laplace's equation in a class of Lipschitz domains. *Comm. Partial Diff. Eqns.*, 19:1217–1233, 1994.

[4] R. M. Brown. The Neumann problem on Lipschitz domains in Hardy spaces of order less than one. *Pac. J. Math.*, 171(2):389–407, 1995.

[5] D. C. Chang, G. Dafni, and E. M. Stein. Hardy spaces, BMO, and boundary value problems for the Laplacian on a smooth domain in $\mathbf{R}^N$. *Trans. Amer. Math. Soc.*, 351(4):1605–1661, 1999.

[6] D.-C. Chang, S. G. Krantz, and E. M. Stein. H^p theory on a smooth domain in $\mathbf{R}^n$ and elliptic boundary value problems. *J. Funct. Anal.*, 114(2):286–347, 1993.

[7] B. Dahlberg. Weighted norm inequalities for the Lusin area integral and nontangential maximal functions for functions harmonic in a Lipschitz domain. *Studia Math.*, 67:297–314, 1980.

[8] B. Dahlberg and C. Kenig. Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains. *Ann. of Math.*, 125:437–466, 1987.

[9] B. E. J. Dahlberg. Estimates of harmonic measure. *Arch. Rational Mech. Anal.*, 65(3):275–288, 1977.

[10] E. Fabes. Layer potential methods for boundary value problems on Lipschitz domains. In *Potential theory—surveys and problems*, volume 1344 of *Lecture notes in math.*, pages 55–80. Springer Verlag, 1988.

[11] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order*. Springer-Verlag, Berlin, 1983.

[12] P. Grisvard. *Elliptic problems in nonsmooth domains*. Pitman, 1985.

[13] D. Jerison and C. Kenig. The Neumann problem on Lipschitz domains. *Bull. Amer. Math. Soc.*, 4:203–207, 1982.

[14] C. E. Kenig. *Harmonic analysis techniques for second order elliptic boundary value problems*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1994.

[15] G. M. Lieberman. Mixed boundary value problems for elliptic and parabolic differential equations of second order. *J. Math. Anal. Appl.*, 113(2):422–440, 1986.

[16] G. M. Lieberman. Optimal Hölder regularity for mixed boundary value problems. *J. Math. Anal. Appl.*, 143(2):572–586, 1989.

[17] J. Pipher and G. Verchota. The Dirichlet problem in L^p for the biharmonic equation on Lipschitz domains. *Amer. J. Math.*, 114(5):923–972, 1992.

[18] F. Rellich. Darstellung der eigenwerte von $\Delta u + \lambda u$ durch ein Randintegral. *Math. Z.*, 46:635–646, 1940.

[19] G. Savaré. Regularity and perturbation results for mixed second order elliptic problems. *Comm. Partial Diff. Eqns.*, 22:869–899, 1997.

[20] J. Serrin and H. F. Weinberger. Isolated singularities of solutions of linear elliptic equations. *Amer. J. Math.*, 88:258–272, 1966.

[21] E. Stein. *Singular integrals and differentiability properties of functions*. Princeton University Press, Princeton NJ, 1970.

[22] E. Stein and G. Weiss. On the theory of harmonic functions in several variables, I. *Acta. Math.*, 103:25–62, 1960.

[23] J. Sykes. *L^p regularity of solutions of the mixed boundary value problem for Laplace's equation on a Lipschitz graph domain*. PhD thesis, University of Kentucky, 1999.

[24] A. Torchinsky. *Real variable methods in harmonic analysis*. Academic Press, 1986.

[25] G. Verchota. Layer potentials and regularity for the Dirichlet problem for Laplace's equation on Lipschitz domains. *J. Funct. Anal.*, 59:572–611, 1984.

Department of Mathematics, University of Kentucky, Lexington, KY 40506
Current address: Department of Mathematics, West Virginia Wesleyan College, Buckhannon, WV 26201
E-mail address: `sykes@wvwc.edu`

Department of Mathematics, University of Kentucky, Lexington, KY 40506
E-mail address: `rbrown@uky.edu`
URL: `http://www.ms.uky.edu/~rbrown`