Estimates for the scattering map associated to a two-dimensional first order system

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Abstract

We consider the scattering transform for the first order system in the plane,

$$\begin{pmatrix} \partial_{\bar{x}} & 0 \\ 0 & \partial_x \end{pmatrix} \psi - \begin{pmatrix} 0 & q^1 \\ q^2 & 0 \end{pmatrix} \psi = 0.$$

We show that the scattering map is Lipschitz continuous on a neighborhood of zero in L^2.

This paper gives an estimate for the scattering map associated to a first-order system

$$D\psi - Q\psi = 0 \tag{1}$$

in the plane. Here, D and Q are defined by

$$D = \begin{pmatrix} \partial_{\bar{x}} & 0 \\ 0 & \partial_x \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & q^1 \\ q^2 & 0 \end{pmatrix}$$

and $\partial_{\bar{x}}$ and ∂_x are the standard derivatives with respect to $x = x^1 + ix^2$ and $\bar{x}$. The entries of the matrix Q, $q^1(x)$ and $q^2(x)$ are complex valued functions on the complex plane. (We will consistently use superscripts to indicate components.) The system (1) was studied by Beals and Coifman [2, 3], and a number of other authors (see Fokas and Ablowitz [7] for an earlier formal treatment, Sung [13, 14, 15] for a detailed rigorous treatment and the review articles [4, 6] for additional references).

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One beautiful feature of the scattering theory as developed by Beals and Coifman is the existence of a non-linear Plancherel identity relating the scattering data S, which is defined below, to the potential Q in (1),

$$\int_{\mathbf{C}} |S|^2 \, d\mu = \int_{\mathbf{C}} |Q|^2 \, d\mu. \tag{2}$$

Here, we use $d\mu$ to denote Lebesgue measure and 2×2-matrices are normed by $|F|^2 = \sum_{j,k=1}^{2} |F^{jk}|^2$. The identity (2) is valid for nice potentials Q satisfying one of the symmetries $Q^* = \pm Q$. Since the map $Q \to S$ is non-linear, the identity (2) does not immediately imply the continuity of this map. The goal of this paper is to provide a proof of the continuity in the L^2-metric of the map $Q \to S$ for small potentials. Our methods apply equally to the inverse map $S \to Q$. We also compute explicitly the size of potentials for which our method works and obtain results for potentials with $\|Q\|_2 < \sqrt{2}$.

Before describing the scattering theory and the proofs of our estimates, we indicate several reasons for interest in this system. The scattering theory for this system can be used to transform the Davey-Stewartson II system to a linear evolution equation. Thus, our estimates can be used to assert that for small initial data, the solution u depends continuously on the initial data in the L^2-norm. However, it is not clear that this method produces a solution for all initial data in a neighborhood of zero in L^2. The application to the Davey-Stewartson system seems to have motivated the work of Beals and Coifman, Fokas and Ablowitz and Sung.

This system also appears in the work of the author and G. Uhlmann [5] who study the inverse conductivity problem in two dimensions and prove that the coefficient is uniquely determined by the Dirichlet to Neumann map for conductivities which have one derivative in L^p, $p > 2$. This is the least restrictive regularity assumption on the coefficient which is known to imply a uniqueness theorem for the coefficient. The argument depends in a crucial way on the non-linear Plancherel identity (2). More recently, Barceló, Barceló and Ruiz [1] have shown that the coefficient depends continuously on the Dirichlet to Neumann map under the *a priori* regularity hypothesis that the coefficient has a gradient which is Hölder continuous. One step of their argument is the observation that the scattering map is continuous if the potential is assumed to be in C^ϵ and compactly supported. The research reported here is perhaps a step towards relaxing this assumption.

We begin the formal development by reviewing the notation of Beals and Coifman's paper [3]. We let $x = x^1 + ix^2$ and $z = z^1 + iz^2$ denote variables in the complex plane. A family of solutions of the free system, $D\psi_0 = 0$, is given by

$$\psi_0(x, z) = \begin{pmatrix} \exp(ixz) & 0 \\ 0 & \exp(-i\bar{x}z) \end{pmatrix}.$$

We look for solutions of (1) of the form $\psi(x, z) = m(x, z)\psi_0(x, z)$ where m approaches the 2×2 identity matrix as $|x| \to \infty$. A computation shows that m must satisfy the equation

$$D_z m = Qm \tag{3}$$

where D_z is the operator $E_z^{-1} D E_z$ and for $z \in \mathbf{C}$, E_z is the map on 2×2 matrix-valued functions defined by

$$E_z f = f^{\mathrm{d}} + A_z^{-1} f^{\mathrm{off}}.$$

Here and below, we use f^{d} and f^{off} to denote the diagonal and off-diagonal parts of the matrix f and $A(x,z) = A_z(x)$ is the matrix given by

$$A(x,z) = \begin{pmatrix} \exp(ix\bar{z} + i\bar{x}z) & 0 \\ 0 & \exp(-ixz - i\bar{x}\bar{z}) \end{pmatrix} = \begin{pmatrix} a^1(x,z) & 0 \\ 0 & a^2(x,z) \end{pmatrix}.$$

The solution to (3) is found by solving the integral equation

$$m = 1 + G_z(Qm). \tag{4}$$

In this equation and below, we use 1 to denote the 2×2 identity matrix. We use $G_z = E_z^{-1} G E_z$ to denote the inverse of D_z, where

$$G(f)(x) = \frac{1}{\pi} \int_{\mathbf{C}} \begin{pmatrix} (x-y)^{-1} & 0 \\ 0 & (\bar{x}-\bar{y})^{-1} \end{pmatrix} f(y)\, d\mu(y).$$

In our situation, with Q small, we can write the solution as a Neumann series, $m = \sum_{k=0}^{\infty} (G_z Q)^k(1)$. Here and throughout this paper, we use Q to denote both the function Q and multiplication operator $f \to Qf$.

Continuing with the scattering theory, we can differentiate the solution with respect to z and obtain the $\partial_{\bar{z}}$-equation

$$\partial_{\bar{z}} m(x,z) = (Tm)(x,z) \tag{5}$$

where the right-hand side Tm is defined by

$$Tm(x,z) = m(x,\bar{z}) S(z) A(x, -\bar{z}). \tag{6}$$

In the definition of T, S is the scattering data which is defined by

$$S(z) = -\frac{1}{\pi} \mathcal{J} \int_{\mathbf{C}} E_z(Q(x) m(x,z))\, d\mu(x) \tag{7}$$

and $\mathcal{J} f = 2(Jf)^{\mathrm{off}}$ with J the 2×2 matrix,

$$J = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

The scattering data S defined in (7) is the function which appears in the left-hand side of (2). The identity (2) is proven for nice functions by Beals and Coifman in [3].

A detailed argument is given by Sung in [13, 14, 15] for potentials Q which are in $L^1 \cap L^\infty$ and much of his work extends to potentials in $L^p \cap L^{p'}$ $(p \neq 2)$.

Our main result is to show that for Q small in L^2, (and say $Q \in \mathcal{S}(\mathbf{C})$, the Schwartz space), the map $Q \to S(Q)$ is Lipschitz continuous in L^2 and that a similar result holds for the inverse map $S \to Q(S)$. These results allow us to extend the maps to a neighborhood of zero in L^2. In fact, the scattering map $Q \to S(Q)$ and the inverse map $S \to Q(S)$ are of the same form. (The more adroit normalization used by Sung makes this clear.) Thus, we concentrate most of our energy on the scattering map.

We remark that the definition of S given in (7) does not make sense if $Q \in L^2$, since I do not know if we can construct m without the assumption of some extra decay on Q. One assumption that will suffice is that $Q \in L^p$ for some $p < 2$.

Our proof is straightforward. We write m, the solution of (4) as a Neumann series and substitute this series into the definition of S, (7). This gives a series of multilinear expressions in Q. We estimate each term and show that we can sum the series. The proof of these estimates depends only on the Hardy-Littlewood-Sobolev theorem, but requires a certain amount of persistence.

The main ingredient is the following well-known result on the first order Riesz potential defined by

$$R(f)(x) = \int_{\mathbf{C}} \frac{f(y)}{|x-y|}\, d\mu(y).$$

We will only consider R acting on scalar functions.

Lemma 1 *Let r and p satisfy $r > 0$ and $1 < p/r < 2$, then the map $f \to [R(|f|^r)]^{1/r}$ satisfies*

$$\|[R(|f|^r)]^{1/r}\|_{\tilde{p}} \leq \alpha(r/p)^{1/r}\|f\|_p$$

where $1/p - 1/\tilde{p} = 1/(2r)$. The constant $\alpha(\theta)$ is bounded for θ in compact subsets of the interval $(1/2, 1)$ and we have

$$\alpha(3/4) = \pi \qquad \textit{and} \qquad \limsup_{\theta \to 3/4} \alpha(\theta) \leq \pi.$$

Proof. According to the Hardy-Littlewood-Sobolev theorem, on fractional integration, (see [12], for example), we have $R : L^s \to L^{\tilde{s}}$ where $1/s - 1/\tilde{s} = 1/2$, provided $1 < s < 2$. We let $\alpha(1/s)$ denote the operator norm of this map on scalar-valued functions.

Now, given p and r as in the Lemma, we define $\tilde{p}$ by $r/p - r/\tilde{p} = 1/2$. We apply the Hardy-Littlewood-Sobolev theorem with exponents p/r and $\tilde{p}/r$ to conclude that

$$\left(\int_{\mathbf{C}} [R(|f|^r)(x)]^{\tilde{p}/r}\, d\mu(x)\right)^{r/\tilde{p}} \leq \alpha(r/p) \left(\int_{\mathbf{C}} |f(x)|^p\, d\mu(x)\right)^{r/p}.$$

Taking the rth root gives the first inequality in the Lemma.

The operator norm for $1/s = 3/4$ was computed by Lieb [9]. The behavior near $\theta = 3/4$ asserted in the Lemma follows from Lieb's result, the boundedness for $1 < p < 2$ and the Riesz-Thorin interpolation theorem. ∎

We now state the main result.

Theorem 2 *Let*

$$N = L^2(\mathbf{C}) \cap \{F : \|F\|_2 < \sqrt{2} \text{ and } F^d = 0\}$$

then the maps $S \to Q$ and $Q \to S$, defined initially on $\mathcal{S}(\mathbf{C}) \cap N$, extend continuously to N. In addition,

$$S \circ Q = Id \quad \text{and} \quad Q \circ S = Id$$

provided Q and $S(Q)$ lie in N (respectively, S and $Q(S)$ lie in N). If $Q = \pm Q^*$, then

$$\int_{\mathbf{C}} |Q|^2 \, d\mu = \int_{\mathbf{C}} |S|^2 \, d\mu.$$

The proof will depend on multi-linear estimates involving repeated fractional integration. For the study of these multi-linear expressions, we will work in a sequence of L^p spaces using the exponents, p_j, s_j and r_j defined by:

$$p_0 = 2 \text{ and } s_0 = 4.$$

Then we define

$$\frac{1}{r_j} = \frac{4}{3p_j}, \qquad j = 0, 1, 2, \ldots \tag{8}$$

$$\frac{1}{p_{j+1}} = \frac{1}{p_j} - \frac{1}{2r_j}, \qquad j = 0, 1, 2, \ldots \tag{9}$$

$$\frac{1}{s_{j+1}} = \frac{1}{s_j} + \frac{1}{2r_j}, \qquad j = 0, 1, 2 \ldots \tag{10}$$

$$\frac{1}{\tilde{s}_j} = \frac{1}{s_j} - \frac{1}{2} \qquad j = 1, 2, \ldots \tag{11}$$

These sequences satisfy

$$\frac{p_j}{r_j} = \frac{4}{3}, \qquad j = 0, 1, 2, \ldots \tag{12}$$

$$\frac{1}{p_j} + \frac{1}{s_j} = \frac{3}{4}, \qquad j = 0, 1, 2, \ldots \tag{13}$$

$$\frac{s_j}{r_j'} = \frac{4}{3}, \qquad j = 0, 1, 2, \ldots \tag{14}$$

$$s_j < 2, \qquad j = 1, 2, \ldots \tag{15}$$

Here and below, we use $r' = r/(r-1)$ to denote the dual exponent. The statement (12) is immediate from (8) and (13) follows from the definitions of p_j and s_j. The third, (14), follows since (13) implies $s_j = 4p_j/(3p_j - 4)$ and $1/r_j' = (3p_j - 4)/(3p_j)$. The last observation (15) follows since $s_1 = 12/7$ and s_j decreases.

The multi-linear expression we will study is $I_k(t, q_0, \ldots q_{2k})$ which is defined for $k = 1, 2, \ldots$ by

$$\begin{aligned} & I_k(t, q_0, \ldots, q_{2k}) \\ = & \int_{\mathbf{C}^{2k+1}} \frac{t(x_0 - x_1 + x_2 - \ldots - x_{2k-1} + x_{2k}) \prod_{j=0}^{2k} q_j(x_j)}{|x_0 - x_1||x_1 - x_2| \cdots |x_{2k-1} - x_{2k}|} \, d\mu(x_0, \ldots x_{2k}). \end{aligned}$$

Also, we will use $I_0(t, q_0) = \int_{\mathbf{C}} t(x) q_0(x) \, d\mu(x)$. The functions t and q_j are scalar-valued. The main estimate for this expression is in the following Lemma.

Lemma 3 *For every $\epsilon > 0$, there exists a constant C_ϵ so that*

$$I_k(t, q_0, \ldots, q_{2k}) \leq C_\epsilon \pi^{2k} (1+\epsilon)^{2k} \|t\|_2 \prod_{j=0}^{2k} \|q_j\|_2.$$

Before, beginning the proof of Lemma 3, we state and prove the following result. The proof of Lemma 3 amounts to applying this result k times.

Lemma 4 *Let $j \geq 0$, $k \geq 1$ and suppose that p_j, s_j, $\tilde{s}_{j+1}$ and r_j are exponents as defined in (8-11). For t and q_j non-negative functions on $\mathbf{C}$, we have*

$$I_k(t, q_0, \ldots, q_{2k}) \leq I_{k-1}(t_1, q_2 \tilde{q}_2, q_3, \ldots, q_{2k})$$

where the functions t_1 and $\tilde{q}_2$ satisfy

$$\begin{aligned} \|t_1\|_{p_{j+1}} & \leq \alpha(3/4)^{1/r_j} \|t\|_{p_j} \\ \|\tilde{q}_2\|_{\tilde{s}_{j+1}} & \leq \alpha(3/4)^{1/r_j'} \alpha(s_{j+1}^{-1}) \|q_1\|_2 \|q_0\|_{s_j}. \end{aligned}$$

Proof. We consider the integral with respect to x_0 in I_k and apply the Hölder inequality using exponents r_j and r_j' to obtain

$$\begin{aligned} & \int_{\mathbf{C}} \frac{t(x_0 - x_1 + x_2 - \ldots + x_{2k}) q_0(x_0)}{|x_0 - x_1|} \, d\mu(x_0) \\ \leq & \left(\int_{\mathbf{C}} \frac{q_0^{r_j'}(x_0)}{|x_0 - x_1|} \, d\mu(x_0) \right)^{1/r_j'} \\ & \quad \times \left(\int_{\mathbf{C}} \frac{t^{r_j}(x_0 - x_1 + x_2 - \ldots + x_{2k})}{|x_0 - x_1|} \, d\mu(x_0) \right)^{1/r_j} \\ = & \tilde{q}_1(x_1) \, t_1(x_2 - x_3 + x_4 - \ldots + x_{2k}). \end{aligned}$$

Here, the function t_1 is defined by

$$t_1(x) = [R(t^{r_j})]^{1/r_j}(x) = \left(\int_{\mathbf{C}} \frac{t^{r_j}(x - w)}{|w|} d\mu(w) \right)^{1/r_j}.$$

One may make the change of variables $w = x_1 - x_0$ in the integral with respect to x_0 to obtain this representation of t_1. The function $\tilde{q}_1$ is defined by $\tilde{q}_1(x_1) = [R(q_0^{r'_j})]^{1/r'_j}(x_1)$. Now, we can rewrite the integral with respect to x_1 in I_k as

$$\int_{\mathbf{C}} \frac{q_1(x_1)\tilde{q}_1(x_1)}{|x_2 - x_1|}\, d\mu(x_1) = R(q_1\tilde{q}_1)(x_2) = R(q_1[R(q_0^{r'_j})]^{1/r'_j})(x_2) \equiv \tilde{q}_2(x_2)$$

which defines $\tilde{q}_2$. Inserting the definitions of t_1 and $\tilde{q}_2$ into I_k gives the inequality relating I_k and I_{k-1}.

We now establish the estimates for t_1 and $\tilde{q}_2$. The estimate

$$\|t_1\|_{p_{j+1}} \leq \alpha(3/4)^{1/r_j}\|t\|_{p_j}$$

follows from Lemma 1 with $p = p_j$ and $r = r_j$. The hypothesis $r_j/p_j \in (1/2, 1)$ follows from (12). To estimate $\tilde{q}_2$, observe that $4/3 < s_{j+1} < 2$ from (13) and (15) and thus we may apply Lemma 1 with $p = s_{j+1}$ and $r = 1$ to obtain the first inequality below. Next, we use Hölder's inequality and Lemma 1 with $p = s_j$ and $r = r'_j$ to obtain the second and third inequalities:

$$\begin{aligned} \|R(q_1[R(q_0^{r'_j})]^{1/r'_j})\|_{\tilde{s}_{j+1}} &\leq \alpha(s_{j+1}^{-1})\|q_1[R(q_0^{r'_j})]^{1/r'_j}\|_{s_{j+1}} \\ &\leq \alpha(s_{j+1}^{-1})\|q_1\|_2\|[R(q_0^{r'_j})]^{1/r'_j}\|_{\tilde{s}_{j+1}} \\ &\leq \alpha(s_{j+1}^{-1})\alpha(3/4)^{1/r'_j}\|q_1\|_2\|q_0\|_{s_j}. \end{aligned}$$

These are the estimates of the Lemma. ∎

We are now ready to give the proof of Lemma 3

Proof of Lemma 3. It suffices to prove the Lemma when all of the functions are positive. We first claim that if $s = 4$ or $4/3$, we have

$$|I_k(t, q_0, \ldots, q_{2k})| \leq C_\epsilon \pi^{2k}(1+\epsilon)^{2k}\|t\|_2\|q_0\|_s\|q_{2k}\|_{s'}\prod_{j=1}^{2k-1}\|q_j\|_2. \tag{16}$$

We use the sequence of exponents defined in (8–11). By Lemma 4 applied k times, we obtain

$$I_k(t, q_0, \ldots, q_{2k}) \leq I_0(t_k, q_{2k}\tilde{q}_{2k})$$

where the function t_k satisfies

$$\|t_k\|_{p_k} \leq \alpha(3/4)^{\sum_{j=0}^{k-1}\frac{1}{r_j}}\|t\|_{p_0}$$

and the function $\tilde{q}_{2k}$ satisfies

$$\|\tilde{q}_{2k}\|_{\tilde{s}_k} \leq \alpha(3/4)^{\sum_{j=0}^{k-1}\frac{1}{r'_j}}\left(\prod_{j=1}^{k}\alpha(s_j^{-1})\right)\prod_{j=1}^{2k-1}\|q_j\|_2\|q_0\|_{s_0}.$$

We write out the integral defining I_0 and obtain

$$I_0(t_k, q_{2k}\tilde{q}_{2k}) = \int_{\mathbf{C}} t_k(x_{2k}) q_{2k}(x_{2k}) \tilde{q}_{2k}(x_{2k})\, d\mu(x_{2k})$$

We have $t_k \in L^{p_k}$, $q_{2k} \in L^{s_0'}$ and $\tilde{q}_{2k} \in L^{\tilde{s}_k}$ where the exponents p_k, $\tilde{s}_k$ and s_0' satisfy

$$\frac{1}{p_k} + \frac{1}{\tilde{s}_k} + \frac{1}{s_0'} = \frac{1}{p_0} + \frac{1}{s_0} - \frac{1}{2} + \frac{1}{s_0'} = 1.$$

Thus Hölder's inequality and the estimates for t_k and $\tilde{q}_{2k}$ imply that

$$|I_k(t, q_0, \ldots, q_{2k})| \leq \alpha(3/4)^k \prod_{j=1}^{k} \alpha(s_j^{-1}) \|q_0\|_4 \|q_{2k}\|_{4/3} \|t\|_2 \prod_{j=1}^{2k-1} \|q_j\|_2.$$

Since $s_j \to 4/3$ as $j \to \infty$, Lemma 1 tells us that $\limsup_{j\to\infty} \alpha(s_j^{-1}) \leq \pi$. Thus, if $\epsilon > 0$, then we obtain (16) with $s = 4$. To obtain the result with $s = 4/3$, we simply need to change variables to replace x_j by x_{2k-j}.

Now that we have the estimate (16), we apply a simple interpolation argument to obtain the Lemma. We define an operator $\mathcal{T}$ by

$$\mathcal{T} q_{2k}(x_0) = \int_{\mathbf{C}^{2k}} \frac{t(x_0 - x_1 + \ldots + x_{2k}) q_1(x_1) \ldots q_{2k}(x_{2k})}{|x_0 - x_1| \ldots |x_{2k-1} - x_{2k}|}\, d\mu(x_1, \ldots, x_{2k}).$$

According to the estimate (16), $\mathcal{T}$ maps $L^4 \to L^4$ and $L^{4/3} \to L^{4/3}$. Hence, by the Riesz-Thorin interpolation theorem, $\mathcal{T}$ is bounded on L^2, with the same bound. This implies the Lemma. ∎

Proof of Theorem 2. We begin by indicating how we construct the solutions m when the potential Q is nice, $Q \in \mathcal{S}(\mathbf{C})$, say, and $\|Q\|_2 < \sqrt{2}$. We can write m as the infinite sum

$$m = 1 + \sum_{j=1}^{\infty} (G_z Q)^j (1). \tag{17}$$

We consider one entry in the matrix $(G_z Q)^j(1)$ with $j = 2k$. Note that for j even, the off-diagonal part $(G_z Q)^j(1)$ is zero. Also, note that the operator E_z involves multiplying by exponentials of modulus 1, and thus the entries of $E_z f$ have the same L^p-norm as the entries of f. We apply Lemma 1 with $p = 4/3$ and $r = 1$ to obtain that

$$\begin{aligned} \|((G_z Q)^{2k})(1)^{11}\|_4 &\leq \pi^{-2k} \|R(Q^{12}(\ldots R(Q^{21})\ldots))\|_4 \\ &\leq (\|Q^{12}\|_2 \|Q^{21}\|_2)^{k-1} \|Q^{12}\|_2 \|Q^{21}\|_{4/3} \\ &\leq \frac{1}{2^{k-1}} \|Q\|_2^{2k-1} \|Q\|_{4/3}. \end{aligned}$$

The last inequality uses the elementary inequality that $\|Q^{12}\|_2 \|Q^{21}\|_2 \leq \frac{1}{2}(\|Q^{12}\|_2^2 + \|Q^{12}\|_2^2)$. A similar argument holds for the remaining entries. Thus the infinite sum in (17) will converge in L^4 if $\|Q\|_2 < \sqrt{2}$. We substitute the sum for m into (7). Since Q and S are off-diagonal, only the even terms of the series for m are needed in the expression for S. Also, note that the operator $G_z = G$ when acting on diagonal matrices. Using these observations and that Q is off-diagonal, we can write

$$\begin{aligned} S(z) &= \frac{-2}{\pi} J \int_{\mathbf{C}} A(x,-z) Q(x) \sum_{j=0}^{\infty} (GQG_zQ)^j(1)\, d\mu(x) \\ &\equiv \frac{-2}{\pi} J \sum_{j=0}^{\infty} S_j(z). \end{aligned}$$

The term with $j = 0$ in this series is essentially the Fourier transform. If we consider one entry, we obtain

$$S_0^{12}(z) = \int_{\mathbf{C}} \exp(-2i(x^1z^1 + x^2z^2)) Q^{12}(x)\, d\mu(x)$$

and we have a similar expression for S_0^{21}. Thus,

$$\frac{1}{\pi^2} \int_{\mathbf{C}} |S_0|^2\, d\mu = \int_{\mathbf{C}} |Q|^2\, d\mu$$

by the standard Plancherel identity. The higher order terms require a bit more work.

In order to simplify the notation, we consider the upper-right entry in S_k

$$\begin{aligned} S_k^{12}(z) &= \frac{1}{\pi^{2k}} \int_{\mathbf{C}^{2k+1}} a^1(-x_0 + x_1 - x_2 + x_3 \ldots + x_{2k-1} - x_{2k}, z) \\ &\quad \times \frac{Q^{12}(x_0)Q^{21}(x_1)\ldots Q^{21}(x_{2k-1})Q^{12}(x_{2k})}{(\bar{x}_0 - \bar{x}_1)(x_1 - x_2)\ldots(\bar{x}_{2k-2} - \bar{x}_{2k-1})(x_{2k-1} - x_{2k})}\, d\mu(x_0, \ldots, x_{2k}). \end{aligned}$$

We take a sufficiently regular (scalar-valued) test function T and write out the expression for S_k^{12} to obtain

$$\begin{aligned} \int_{\mathbf{C}} T(z) S_k^{12}(z)\, d\mu(z) &= \frac{1}{\pi^{2k}} \int_{\mathbf{C}^{2k+2}} T(z) a^1(-x_0 + x_1 - x_2 + x_3 \ldots + x_{2k-1} - x_{2k}, z) \\ &\quad \times \frac{Q^{12}(x_0)Q^{21}(x_1)\ldots Q^{21}(x_{2k-1})Q^{12}(x_{2k})}{(\bar{x}_0 - \bar{x}_1)(x_1 - x_2)\ldots(\bar{x}_{2k-2} - \bar{x}_{2k-1})(x_{2k-1} - x_{2k})}\, d\mu(z, x_0, \ldots, x_{2k}). \end{aligned}$$

Note that if each of the functions T and Q are in $L^1 \cap L^\infty$, say, then the integral on the right of the previous expression converges absolutely. Thus, we may use the Fubini theorem to carry out the integration with respect to z first and obtain the expression

$$\begin{aligned} &\frac{1}{\pi^{2k}} \int_{\mathbf{C}^{2k+1}} \hat{T}(2(x_0 - x_1 + x_2 - \ldots - x_{2k-1} + x_{2k})) \\ &\quad \times \frac{Q^{12}(x_0)Q^{21}(x_1)\ldots Q^{21}(x_{2k-1})Q^{12}(x_{2k})}{(\bar{x}_0 - \bar{x}_1)(x_1 - x_2)\ldots(\bar{x}_{2k-2} - \bar{x}_{2k-1})(x_{2k-1} - x_{2k})}\, d\mu(x_0, \ldots, x_{2k}). \end{aligned}$$

Here, $\hat{T}$ is the Fourier transform of T and is defined by $\hat{T}(z) = \int f(x) e^{-\operatorname{Re} x\bar{z}}\, d\mu(x) = \int f(x) a^1(-\frac{1}{2}x, z)\, d\mu(z)$. Thus we have that

$$\left| \int_{\mathbf{C}} T(z) S^{12}(z)\, d\mu(z) \right| \leq \frac{1}{\pi^{2k}} I_k(|\hat{T}(2\cdot)|, |Q^{12}|, |Q^{21}|, \ldots, |Q^{12}|).$$

Now, Lemma 3 implies for each $\epsilon > 0$, there is a constant C_ϵ so that

$$\|S_k\|_2 \leq C_\epsilon \frac{(1+\epsilon)^{2k}}{2^k} \|Q\|_2^{2k}$$

and hence the series

$$\sum_{k=0}^{\infty} S_k$$

converges in L^2 if $\|Q\|_2 < \sqrt{2}$.

We now turn to the proof that the map $Q \to S$ is Lipschitz continuous. Let Q and $\tilde{Q}$ be two matrix potentials. We write $S = S(Q)$, then we have

$$\begin{aligned} &\left| \int_{\mathbf{C}} T(z) (S_k^{12}(Q)(z) - S_k^{12}(\tilde{Q})(z))\, d\mu(z) \right| \\ &\leq \frac{1}{\pi^{2k}} \sum_{j=0}^{2k} I_k(|\hat{T}(2\cdot)|, |Q^{12}|, |Q^{21}|, \ldots, |Q^{ab} - \tilde{Q}^{ab}|, \ldots, |\tilde{Q}^{12}|) \end{aligned}$$

where the difference occurs in the jth spot and $ab = 12$ if j is even or 21 if j is odd. Thus, Lemma 3 implies that if $\|Q\|_2 < M$ and $\|\tilde{Q}\|_2 < M$, then for each $\epsilon > 0$, there is a constant C_ϵ so that

$$\|S^{12}(Q) - S^{12}(\tilde{Q})\|_2 \leq C_\epsilon \|Q - \tilde{Q}\|_2 M^2 \sum_{k=0}^{\infty} (2k+1)(1+\epsilon)^{2k} \frac{1}{2^{k-1}} M^{2(k-1)}.$$

Since we have $M < \sqrt{2}$, we have the Lipschitz continuity also.

The inverse map is handled similarly so we only give a sketch of the argument. The map which takes Q to S is given by

$$Q(x) = \frac{1}{\pi} \mathcal{J} \int_{\mathbf{C}} Tm(x,z)\, d\mu(z) \tag{18}$$

where the map T depends on S and was defined in (6). We have that m may be represented as

$$m = (I - CT)^{-1}(1)$$

where C is the Cauchy transform

$$Cf(z) = \frac{1}{\pi} \int_{\mathbf{C}} \frac{f(w)}{z - w}\, d\mu(w)$$

acting on matrix valued functions. As before, we can write m as the series,

$$m = \sum_{j=0}^{\infty} (CT)^j(1) \tag{19}$$

Substituting (19) into (18) gives a series representation for Q, $Q(x) = \frac{1}{\pi}\sum_{k=0}^{\infty} Q_k(x)$ where the kth term is given by

$$\begin{aligned} Q_k^{12}(x) &= \frac{1}{\pi^{2k}} \int_{\mathbf{C}^{2k+1}} \frac{S^{12}(z_0)S^{21}(z_1)\ldots S^{12}(z_{2k})}{(\bar z_0 - z_1)(\bar z_1 - z_2)\ldots(\bar z_{2k-1} - z_{2k})} \\ &\qquad \times a^1(x, z_0 - \bar z_1 + \ldots - \bar z_{2k-1} + z_{2k})\, d\mu(z_0, \ldots z_{2k}). \end{aligned}$$

Arguing as before, we can show that for each $\epsilon > 0$, there is constant C_ϵ so that

$$\|Q_{2k}\|_2 \le C_\epsilon \frac{(1+\epsilon)^{2k}}{2^k} \|S\|_2^{2k+1}$$

and also obtain that Q depends continuously on S.

The statements that $Q \circ S = Id$ and $S \circ Q = Id$ follow from continuity and the corresponding results for nice potentials as proved by Sung [13, 14, 15]. However, one must replace Lemma 2.3 of [13]. The uniqueness of m needed to carry Sung's arguments holds for $\|Q\|_2 < \sqrt{2}$, thanks to the estimate for $\alpha(3/4)$ in Lemma 1 (see also (17)). Also, the extension of the non-linear Plancherel identity to functions in L^2 is immediate from the result for smooth functions and the continuity of the map $Q \to S$. ∎

We give an application of our estimate. This application depends on the fact that we have shown that the scattering map and its inverse map a neighborhood of 0 in L^2 into L^2. Since the domain and range lie in the same space, L^2, it is particularly simple to construct the solution.

Recall that the Davey-Stewartson II equation as studied by Beals and Coifman is the evolution equation in two space dimensions given by

$$\begin{cases} q_t = iq_{x^1x^2} - 4irq \\ r_{x^1x^1} + r_{x^2x^2} = (|q|^2)_{x^1x^2} \end{cases}$$

If we set

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \bar q(x,t) & 0 \end{pmatrix}$$

and apply the scattering map to obtain $S((\cdot,t) = S(Q)(\cdot,t)$, then the function $S(t)$ satisfies

$$S_t = (4iz^1z^2)S.$$

Thus the solution of the Davey-Stewartson II equation is the upper-right entry of the matrix

$$Q(\cdot,t) = Q(\exp(4itz^1z^2)S(Q(\cdot,0))). \tag{20}$$

This is established for nice potentials Q by Beals and Coifman [3], see also Sung for a detailed argument [13, 14, 15]. Our contribution is the observation that the maps Q and S are continuous in L^2 which immediately implies the following result. See Ghidaghlia and Saut [8] and Linares and Ponce [10] for additional results on the Davey-Stewartson systems.

Corollary 5 *The scattering solution of the Davey-Stewartson II system as defined in (20) gives a solution which depends continuously on the initial data in L^2. If the initial data $q_j(\cdot,0)$ satisfies $\|q_j(\cdot,0)\|_2 < 1$ and say $q_j(x,0) \in \mathcal{S}(\mathbf{C})$ for $j=1$ and 2, then the solutions $q_j(x,t)$ satisfy*

$$\|q_1(\cdot,t) - q_2(\cdot,t)\|_2 \leq C\|q_1(\cdot,0) - q_2(\cdot,0)\|_2.$$

We close with three questions related to the above results.

1. Can we remove the smallness restriction from our Theorem 2? It is known that if one of $Q = \pm Q^*$ holds and $Q \in L^2$, then the operator $I - G_z Q$ is invertible. This fact does not seem to be in the literature. To see the invertibility of $I - G_z Q$ on $L^{4/3}$, we observe that arguments similar to those of Nachman [11, Lemma 4.2] show that the map $f \to G_z Q f$ is compact on $L^{4/3}$. The injectivity of the map $f \to f - G_z Q f$ when $Q = \pm Q^*$ follows from Corollary 3.11 of [5]. This Corollary is an observation of Nachman's. Hence, the Fredholm theory gives the claim.

2. Can we construct the Jost solutions m when Q is only assumed to be in L^2?

3. The expression (20) is defined, if Q is just in L^2 and small. In what sense does this expression solve the Davey-Stewartson equations?

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