Uniqueness in the inverse conductivity problem for conductivites with 3/2 derivatives in L^p, $p > 2n$

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The purpose of this note is to establish a small extension of a result of Panchenko, Päivärinta and Uhlmann [14]. These authors recently showed that we have uniqueness in the inverse conductivity problem for conductivities which are in the class $C^{3/2}$ in three dimensions and higher. This built on earlier work of one the authors, Brown [3]. In this note, we relax this condition to conductivities which have 3/2 derivatives in L^p for $p > 2n$. We will obtain an end-point result with $p = 2n$ for a related problem for Schrödinger equations. However, the problem of defining the trace on the boundary prevents us from obtaining uniqueness for the inverse conductivity problem at the end-point. Our results are in $\mathbf{R}^n$ with $n \geq 3$. Better results are available in $\mathbf{R}^2$ in the work of Brown and Uhlmann [4].

The inverse conductivity problem is the problem of determining the coefficient γ in the elliptic operator $\mathrm{div}\gamma\nabla$ from information about solutions to this operator at the boundary. This problem was posed in the mathematics community by A.P. Calderón [6]. As has been known for some time [8, 12, 16], the key technique in establishing such uniqueness results is the construction of solutions which are asymptotic to harmonic exponentials at infinity. To construct these solutions, we use the standard relationship between an elliptic operator of the form $\mathrm{div}\gamma\nabla$ and a Schrödinger operator. Of course, the coefficient γ is a scalar valued function. The Schrödinger operators we consider are of the form

$$\Delta - q$$

where the potential $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$ may be a distribution if γ is not smooth. We look for solutions v of the Schrödinger equation $\Delta v - qv = 0$ which are asymptotic to $e^{x\cdot\zeta}$ where ζ is in $\mathbf{C}^n$ and satisfies $\zeta \cdot \zeta = \sum_j \zeta_j^2 = 0$. A simple calculation shows that if we look for solutions which are of the form

$$v(x, \zeta) = e^{x\cdot\zeta}(1 + \psi(x, \zeta))$$

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then the function ψ must satisfy

$$\Delta\psi + 2\zeta\cdot\nabla\psi - q\psi = q.$$

And, at least formally, the solution ψ can be constructed as the series

$$\psi(x,\zeta) = \sum_{j=0}^{\infty}(G_\zeta m_q)^j G_\zeta(q). \tag{1}$$

In this expression, m_q is the multiplication operator given by q

$$m_q(\phi) = q\phi$$

and G_ζ is the inverse of $\Delta + 2\zeta\cdot\nabla$ defined by

$$\widehat{G_\zeta(f)}(\xi) = \frac{\hat{f}(\xi)}{-|\xi|^2 + 2i\zeta\cdot\xi}.$$

Since the symbol $-|\xi|^2 + 2i\zeta\cdot\xi = 0$ vanishes to first order on a sphere of codimension 2, the right-hand side of the definition of G_ζ will be a tempered distribution if f is in the Schwartz class. The Fourier transform is normalized by $\hat{f}(\xi) = \int f(x)e^{-ix\cdot\xi}\,dx$. To make sense of the infinite sum (1), we need to make estimates for the operator G_ζ and the multiplication operator m_q acting on Banach spaces. The contribution of this note beyond previous work is to provide an improved estimate for m_q. To be precise, we will show that if q is in the Sobolev space $W^{-s,n/s}$, then the multiplication operator m_q (which is defined, say, as a map from $\mathcal{S}$ to $\mathcal{S}'$) also maps $W^{s,2}$ to $W^{-s,2}$. The case $s=1$ is a consequence of such familiar results as the product rule and the Sobolev embedding theorem. To see this, suppose $u, v \in W^{1,2}$, then we have that uv lies not only in $W^{1,1}$, but we also have $\nabla(uv) \in L^{n/(n-1)}$. This follows from the product rule and Sobolev embedding: $u\nabla v + v\nabla u$ is in $L^{n/(n-1)}$ since $\nabla v, \nabla u \in L^2$ and $u, v \in L^{2n/(n-2)}$. Thus, we have

$$|\langle\frac{\partial p}{\partial x_j}, uv\rangle| = \left|\int p\frac{\partial uv}{\partial x_j}\,dx\right| \le C\|p\|_{L^n}\|u\|_{W^{1,2}}\|v\|_{W^{1,2}}$$

from which it follows that the map $u \to u\frac{\partial p}{\partial x_j}$ maps $W^{1,2}$ to the dual space $W^{-1,2}$ when p is in L^n. Here, we use $\langle\cdot,\cdot\rangle$ to denote the bilinear pairing between distributions and functions.

In this paper, we contribute nothing to the analysis of G_ζ. The estimates used are from the paper of Sylvester and Uhlmann [16]. It is possible that some improvement can be made here. We expect that one should be able to prove uniqueness for conductivities which have 3/2 derivatives in L^p with $p > 2n/3$. However, the straightforward generalization of the argument presented below would require that $f \to \nabla G_\zeta f$ map L^p functions which are compactly supported to functions which are locally in L^r with

p and r satisfying $1/p - 1/r = 1/n$. Many such estimates fail, see [2] for further discussion.

Finally, let us mention that one of us (Brown) conjectures that uniqueness should hold when the conductivity γ has one derivative in L^p with $p > n$. The methods presented here do not come close to addressing this conjecture.

We begin the formal development by stating a theorem on products of functions in Sobolev spaces $W^{s,p}(\mathbf{R}^n)$. There does not seem to be a standard notation for these spaces. For $s \in \mathbf{R}$ and $1 < p < \infty$, we will use $W^{s,p}$ to denote the space of distributions (or functions, if $s \geq 0$) which are defined by the Bessel potential operator. Thus $u \in W^{s,p}$ if and only if $J_{-s}u \in L^p$ where J_s is the operator given by

$$f \to ((1+|\xi|^2)^{-s/2}\hat{f}(\xi))\check{}.$$

We recall that if $s = 0$ and $1 < p < \infty$, then this space coincides with L^p. If s is a positive integer and $1 < p < \infty$, then the space $W^{s,p}$ is precisely the functions with s derivatives in L^p. For $1 < p < \infty$, this space is also known as the Triebel-Lizorckin space $F_2^{s,p}$. We refer to the monograph of Triebel [18] for properties of these spaces.

Theorem 1 *Let $u \in W^{s,p}$ and let $v \in W^{s,q}$, with $1 < p, q < \infty$, $1/p + 1/q \leq 1$, and $0 \leq s < n\min(1/p, 1/q)$. Then $uv \in W^{s,r^*}$ where $1/r^* = 1/p + 1/q - s/n$.*

This result is well-known, see the monograph of Runst and Sickel [15, p. 177]. This theorem immediately implies the following Corollary.

Corollary 2 *If $q \in W^{-1/2,2n}$ with $n \geq 3$, then the operator m_q given by $m_q(u) = qu$ maps $W^{1/2,2} \to W^{-1/2,2}$ and satisfies*

$$\|m_q(u)\|_{W^{-1/2,2}} \leq C\|u\|_{W^{1/2,2}}\|q\|_{W^{-1/2,2n}}.$$

Proof. To be precise, $m_q(u)$ is defined by $\langle m_q(u), v\rangle = \langle q, uv\rangle$. Using the duality between $W^{s,n/(n-s)}$ and $W^{-s,n/s}$ which is valid (at least) for $1 < n/s < \infty$, and Theorem 1 we conclude that

$$|\langle q, vu\rangle| \leq C\|q\|_{W^{-1/2,2n}}\|uv\|_{W^{1/2,2n/(2n-1)}} \leq C\|q\|_{W^{-1/2,2n}}\|u\|_{W^{1/2,2}}\|v\|_{W^{1/2,2}}.$$

This inequality implies the Corollary. ∎

Now, we recall standard estimates for the operator G_ζ. In order to simplify what follows, we will not use the weighted estimates of Sylvester and Uhlmann, but a simple consequence of these estimates. This will allow us to avoid mention of weighted Sobolev spaces. A weighted version of the previous Corollary is undoubtedly true, but it might be a chore to chase down the proof. We will restrict our attention to compactly supported potentials. This restriction is acceptable for our application to the inverse conductivity problem. However, there may be interest in Schrödinger operators where the potential is not compactly supported. If X is a space of distributions, we will fix $R_0 > 1$ and let X_c denote the subspace of X which is supported in $\bar{B}_{R_0}(0)$. Our notation obscures the dependence of the space X_c on R_0, however we will give the dependence of the constants in our estimates on R_0. Also, we are using $B_s(x)$ to denote the ball of radius s and center x. The following result is due to Sylvester and Uhlmann [16]. The spaces we use are similar to those used by Agmon and Hörmander (see [10, p. 227]), except that we require smoothness of order s. We fix η an infinitely differentiable function which is supported in a ball of radius 1 and is identically one in a neighborhood of $\bar{B}_{1/2}(0)$ and set $\eta_R(x) = \eta(x/R)$. With this function η, we define the space B^*_s to be the space of distributions for which the norm

$$\|f\|_{B^*_s} = \sup_{R>1} R^{-1/2} \|\eta_R f\|_{W^{s,2}}$$

is finite. We leave it as an exercise to see that different choices of η, subject to the above conditions, give equivalent norms on B^*_s.

Theorem 3 *If $n \geq 3$, $|\zeta| \geq 1$, $\zeta \cdot \zeta = 0$ and $0 \leq s \leq 1$, then*

$$\|G_\zeta f\|_{B^*_{1/2}} \leq \frac{C R_0^{1/2}}{|\zeta|^s} \|f\|_{W_c^{-1/2+s,2}}.$$

Proof. From the main estimate in [16] (see also [3] for the gradient estimate), we find that

$$\int_{\mathbf{R}^n} (1+|x|^2)^{-1/2} (|G_\zeta f(x)|^2 |\zeta|^2 + |\nabla G_\zeta f(x)|^2)\, dx \leq C \int_{\mathbf{R}^n} (1+|x|^2)^{1/2} |f(x)|^2\, dx. \quad (2)$$

On the ball $B_R(0)$, (with $R > 1$) the weight on the left-hand side of (2) is bounded below by $2R$ and the weight on the right-hand side is bounded above by $2R$. This gives the estimate

$$R^{-1/2} \|G_\zeta f\|_{L^2(B_R(0))} \leq \frac{C}{|\zeta|} R_0^{1/2} \|f\|_{L^2_c}. \quad (3)$$

Also, since G_ζ and ∇ commute, we obtain that for f compactly supported in B_{2R_0}

$$R^{-1/2} \|\nabla G_\zeta f\|_{L^2(B_R(0))} \leq \frac{C}{|\zeta|} R_0^{1/2} \|\nabla f\|_{L^2_c}. \quad (4)$$

We let η be a smooth function as in the definition of the $B^*_\cdot$-spaces. The inequalities (3) and (4) together imply that for $s = 0, 1$, we have

$$R^{-1/2} \|\eta_R G_\zeta \eta_{2R_0} f\|_{W^{s,2}} \leq \frac{C}{|\zeta|} R_0^{1/2} \|f\|_{W^{s,2}}. \quad (5)$$

Complex interpolation implies that the inequality (5) continues to hold for $0 < s < 1$. Finally, if we use the estimate for the gradient in (2), it is not hard to see that for $R, R_0 > 1$, we have

$$R^{-1/2} \|\eta_R G_\zeta \eta_{2R_0} f\|_{W^{s+1,2}} \leq C R_0^{1/2} \|f\|_{W^{s,2}} \quad (6)$$

for $s = 0$. By duality (and interchanging R and $2R_0$), we obtain (6) with $s = -1$ also. Then, interpolating gives (6) with $s = -1/2$. Interpolating between (6) with $s = -1/2$ and (5) with $s = 1/2$, and taking the supremum in R gives the conclusion of the theorem. Notice that if f is in $W^{s,2}_c$ and $\eta = 1$ in a neighborhood of $\bar{B}_{1/2}(0)$, then $\eta_{2R_0} f = f$, hence the estimates for G_ζ on $W^{s,2}_c$ follow from estimates for the map $f \to G_\zeta(\eta_{2R_0} f)$. ∎

Now, we consider a potential q in $W^{-1/2,2n}_c$ and consider the equation

$$(\Delta + 2\zeta \cdot \nabla)\psi - q\psi = f. \tag{7}$$

In what follows, we will use $G_{\zeta,q}$ to denote the inverse of the operator $\Delta + 2\zeta \cdot \nabla - q$. We prove the existence of this map by showing that it maps from $W^{-1/2,2}_c$ to the space $B^*_{1/2}$.

Theorem 4 *Let $q \in W^{-1/2,2n}_c$, then there exists $C = C_0(q)$ so that for $|\zeta| \geq C_0(q)$, we may find a unique solution, $\psi = G_{\zeta,q} f$, to (7) in $B^*_{1/2}$. This map satisfies*

$$\|G_{\zeta,q} f\|_{B^*_{1/2}} \leq A\|f\|_{W^{-1/2,2}_c}.$$

The constant $A = A(R_0)$ in the previous estimate is independent of q and ζ.

Furthermore, we have

$$\lim_{|\zeta|\to\infty} \|G_{\zeta,q} f\|_{B^*_{1/2}} = 0.$$

To carry out the proof of the Theorem, we begin with the case of smooth potentials. The key point is that smooth potentials are dense in $W^{-1/2,2n}$ and that for smooth potentials, the norm of the inverse does not depend on the size of q. The size of q only enters in determining how large ζ must be.

Lemma 5 *Suppose $q \in C^\infty_c(\mathbf{R}^n)$ and that $\operatorname{supp} q \subset B_{R_0}(0)$. Then there is a constant $C_0 = C_0(q)$ so that for $|\zeta| > C_0(q)$ and $0 \leq s \leq 1$, $G_{\zeta,q}$ satisfies*

$$\|G_{\zeta,q} f\|_{B^*_{1/2}} \leq \frac{A}{|\zeta|^s}\|f\|_{W^{-1/2+s,2}_c}. \tag{8}$$

The constant $A = A(R_0)$ is independent of ζ and q.

Proof. We write

$$G_{\zeta,q}(f) = \sum_{j=0}^{\infty} (G_\zeta m_q)^j G_\zeta f.$$

According to Theorem 3, we have

$$\|G_\zeta f\|_{B^*_{1/2}} \leq \frac{C R_0^{1/2}}{|\zeta|^s}\|f\|_{W^{-1/2+s,2}}$$

for $0 \leq s \leq 1$. Because, q is smooth and $\eta = 1$ in a neighborhood of $B_{1/2}(0)$, it is clear that
$$\|m_q f\|_{W_c^{1/2,2}} \leq C(q)\|\eta_{2R_0} f\|_{W^{1/2,2}} \leq C(q) R_0^{1/2} \|f\|_{B^*_{1/2}}.$$
Using this and the estimate in Theorem 3 with $s = 1$, we obtain

$$\|G_\zeta m_q f\|_{B^*_{1/2}} \leq \frac{C(q)R_0}{|\zeta|}\|f\|_{B^*_{1/2}}.$$

Now, if we require $|\zeta|$ to be sufficiently large, the norm of $G_\zeta \circ m_q$ on $B^*_{1/2}$ will be bounded by 1/2 and hence the norm of $\sum_j (G_\zeta \circ m_q)^j$ as an operator on $B^*_{1/2}$ will be at most 2. This proves the Lemma. ∎

Proof of Theorem 4. We suppose now that q is in $W^{-1/2,2n}$ and is supported, say, in $B_{R_0/2}(0)$. We let $\epsilon > 0$, be a number to be determined later. We can write $q = q_s + q_r$ where the smooth part of q, q_s, is in $C_0^\infty(\mathbf{R}^n)$ and is supported in $B_{R_0}(0)$ and the remainder, the rough part, q_r is also supported in $B_{R_0}(0)$ and satisfies $\|q_r\|_{W^{-1/2,2n}} < \epsilon$. We suppose $f \in B^*_{1/2}$ and observe that with η as in the definition of $B^*_{1/2}$, we have that $m_{q_r}(f) = m_{q_r}(\eta_{2R_0} f)$ because $\eta_{2R_0} = 1$ on the support of q_r. We use the estimate for G_{ζ,q_s} on $W_c^{-1/2,2}$ in Lemma 5 and the estimate for m_{q_r} in Corollary 2 to obtain

$$\begin{aligned} \|G_{\zeta,q_s} m_{q_r} f\|_{B^*_{1/2}} &\leq A\|m_{q_r} f\|_{W_c^{-1/2,2}} \\ &\leq CA\|q_r\|_{W^{2n,-1/2}}\|\eta_{2R_0} f\|_{W^{1/2,2}} \\ &\leq CA\epsilon\sqrt{2R_0}\|f\|_{B^*_{1/2}}. \end{aligned} \tag{9}$$

We search for u which satisfies $\Delta u + 2\zeta \cdot \nabla u - q_s u - q_r u = f$ and if we let G_{ζ,q_s} be an inverse to $\Delta u + 2\zeta \cdot \nabla - q_s$, then we can write

$$G_{\zeta,q} f = \sum_{j=0}^{\infty} (G_{\zeta,q_s} m_{q_r})^j G_{\zeta,q_s} f. \tag{10}$$

If we choose ϵ so that $CA\epsilon\sqrt{2R_0} = 1/2$, then the series in (10) will converge thanks to the estimate in (9). According to Lemma 5, the maps G_{ζ,q_s} will exist for $|\zeta|$ sufficiently large depending on q_s and hence on q.

To obtain uniqueness, we suppose that $u \in B^*_{1/2}$ is a distribution solution of $\Delta u + 2\zeta \cdot \nabla u - qu = 0$ in $\mathbf{R}^n$. Note that u must have half of a derivative for the product qu to be defined. Again, we split $q = q_r + q_s$ and by Proposition 2.1 in Sylvester and Uhlmann [16], see also Hörmander [9, Theorem 7.1.27], there is a unique solution of $\Delta u + 2\zeta \cdot \nabla u - q_s u = q_r u$, namely $u = G_{\zeta,q_s}(q_r u)$. By the estimate of Lemma 5, we obtain

$$\|u\|_{B^*_{1/2}} \leq A\|q_r\|_{W^{-1/2,2n}}\|u\|_{B^*_{1/2}},$$

at least for $|\zeta|$ sufficiently large. Since the norm of q_r may be arbitrarily small, it follows that we have $u = 0$ for ζ large.

Finally, we observe that if we fix $f \in W^{-1/2,2}$, then we have

$$\lim_{|\zeta|\to\infty} \|\eta_R G_{\zeta,q} f\|_{B^*_{1/2}} = 0.$$

To see this, we let $\epsilon > 0$ and split $f = f_s + f_r$ with $\|f_r\|_{W^{-1/2,2}} < \epsilon$. Let q_s and q_r be the splitting used above in constructing $G_{\zeta,q}$. Using the representation for $G_{\zeta,q}$, (10), and the estimate for G_{ζ,q_s}, (8), we have

$$\|G_{\zeta,q} f_s\|_{B^*_{1/2}} \le C(f_s, q)/|\zeta|. \tag{11}$$

While the estimate of this theorem, which we have already proved, shows that for $|\zeta|$ sufficiently large, we have

$$\|G_{\zeta,q} f_r\|_{B^*_{1/2}} \le A\epsilon. \tag{12}$$

Together these observations imply that $\limsup_{|\zeta|\to\infty} \|G_{\zeta,q} f\|_{B^*_{1/2}} < A\epsilon$ and since ϵ is an arbitrary positive number, the limit is zero. ∎

Corollary 6 *If $q \in W^{-1/2,2n}_c$, then there is a constant $C_0(q)$ so that if ζ satisfies $|\zeta| > C_0(q)$ and $\zeta \cdot \zeta = 0$, we can construct solutions v of $\Delta v - qv = 0$ of the form $v(x,\zeta) = e^{x\cdot\zeta}(1 + \psi(x,\zeta))$ with*

$$\lim_{|\zeta|\to\infty} \|\psi\|_{B^*_{1/2}} = 0$$

Furthermore, ψ is the unique function in $B^*_{1/2}$ and so that $v = e^{x\cdot\zeta}(1+\psi(x,\zeta))$ satisfies $\Delta v - qv = 0$.

Proof. We construct $\psi = \psi(\cdot,\zeta) = G_{\zeta,q}(q)$ where $G_{\zeta,q}$ was constructed in Theorem 4. The existence and the limiting behavior of ψ follow from Theorem 4. It is a simple calculation to see that the ψ and then the v constructed in this way have the desired properties.

If we have two such ψ, call them ψ_1 and ψ_2, then their difference is a solution of the homogeneous equation

$$\Delta(\psi_1 - \psi_2) + 2\zeta \cdot \nabla(\psi_1 - \psi_2) - q(\psi_1 - \psi_2) = 0.$$

According to Theorem 4, 0 is the only solution of this equation in $B^*_{1/2}$. ∎

In what follows, we will also need function spaces on domains. For $s \ge 0$, we define the space $W^{s,p}(\Omega)$ as the image of the space $W^{s,p}(\mathbf{R}^n)$ under the map $u \to u|_\Omega$. Thus elements in the space $W^{s,p}(\Omega)$ automatically have extensions to all of $\mathbf{R}^n$. We also recall that the space $W^{s,p}_0(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in the space $W^{s,p}(\Omega)$.

Next, we recall the definition of the Dirichlet to Neumann map, which we denote Λ_γ, for the elliptic operator $\operatorname{div}\gamma\nabla$. We suppose that we have a bounded domain with Lipschitz boundary. We define the space $W^{1/2,2}(\partial\Omega)$ as the quotient space

$W^{1,2}(\Omega)/W_0^{1,2}(\Omega)$. It is well-known that this space can be identified with the Besov space $B_2^{1/2,2}$ on the boundary. The coefficient γ in the operator $\mathrm{div}\gamma\nabla$ will lie in $L^\infty(\Omega)$ and satisfy

$$\delta \leq \gamma \leq 1/\delta \tag{13}$$

for some $\delta > 0$. If f is in $W^{1,2}(\Omega)$, we can solve the Dirichlet problem

$$\begin{cases} \mathrm{div}\gamma\nabla u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

This solution u is, of course, independent of the particular representative f of an equivalence class in $W^{1/2,2}(\partial\Omega)$. Given the solution u of the Dirichlet problem with data f, we may define a map $\Lambda_\gamma : W^{1/2,2}(\partial\Omega) \to W^{-1/2,2}(\partial\Omega)$ by

$$\Lambda_\gamma(f)(g) = \int_\Omega \gamma\nabla u \cdot \nabla g \, dx.$$

Since u is a solution, the expression on the right does not change if we add a function in $W_0^{1,2}(\Omega)$ to g. Thus, we may define $\Lambda_\gamma(f)$ as an element of the dual of $W^{1/2,2}(\partial\Omega)$.

Now, we quote the following result on determining the coefficient at the boundary. This will be an important step in connecting the Dirichlet to Neumann map in Ω to a problem in all of space. In our argument below, we will work with two conductivities γ_1 and γ_2 with $\sqrt{\gamma_j}$ in $W^{3/2,2n+\epsilon}(\Omega)$ and try to extend them to preserve smoothness and so that the extensions are equal in the complement of Ω. The next two results explain when this can be done.

Proposition 7 *Assume that $\partial\Omega$ is Lipschitz and assume that for some $\epsilon > 0$, γ_1 and γ_2 are in $W^{3/2,2n+\epsilon}$ and that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then we have $\gamma_1 = \gamma_2$ and $\nabla\gamma_1 = \nabla\gamma_2$ on $\partial\Omega$.*

Proof. The hypothesis that γ_j is in $W^{3/2,2n+\epsilon}(\Omega)$ implies, by the Sobolev embedding theorem, that there is an $\epsilon' > 0$ so that $\nabla\gamma_j$ is in $C^{1+\epsilon'}(\bar{\Omega})$ for $j = 1, 2$. It has been proven many times that the boundary values of a conductivity and its derivatives are determined by the Dirichlet to Neumann map. The result stated here for $C^{1+\epsilon'}$ conductivities may be found in Alessandrini [1]. ∎

Corollary 8 *If, for some $\epsilon > 0$, $\sqrt{\gamma_1}$ and $\sqrt{\gamma_2}$ are in $W^{3/2,2n+\epsilon}(\Omega)$, and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then we may extend γ_1 and γ_2 to all of $\mathbf{R}^n$ so that $\sqrt{\gamma_j} - 1 \in W^{3/2,2n+\epsilon}$ and $\gamma_1 = \gamma_2$ in $\mathbf{R}^n \setminus \Omega$.*

Proof. According to the previous proposition, $\gamma_1 = \gamma_2$ on $\partial\Omega$ and also $\nabla\gamma_1 = \nabla\gamma_2$ on $\partial\Omega$. By our definition of the space $W^{s,p}(\Omega)$, it is immediate that there are extensions of $\sqrt{\gamma_1}$ and $\sqrt{\gamma_2}$ to all of $\mathbf{R}^n$. Using smooth cutoff functions, we can arrange that the extensions (which we still denote by $\sqrt{\gamma_j}$) have $\sqrt{\gamma_j} - 1$ in $W_c^{3/2,2n+\epsilon}(\mathbf{R}^n)$ for $j = 1, 2$. We claim that the function

$$\beta = \begin{cases} \sqrt{\gamma_2} - \sqrt{\gamma_1}, & \text{in } \Omega, \\ 0, & \text{in } \bar{\Omega}^c, \end{cases}$$

lies in $W^{3/2,2n+\epsilon}(\mathbf{R}^n)$. This will suffice to prove our Corollary since $\sqrt{\gamma_1}$ and $\sqrt{\gamma_1} + \beta$ provide the two extensions which agree in the complement of Ω.

Thus, we must establish the claim. Of course, this depends on the hypothesis that γ_1 and γ_2 agree to first order on the boundary of Ω. In fact, this claim is a consequence of Corollary 2.11 in Triebel's monograph [18, p. 210]. Alert readers will note that this monograph assumes that the domain is smooth. However, that assumption is not needed when $p > 1$. The key step is to show that the characteristic function of Ω is multiplier on $W^{s,p}(\mathbf{R}^n)$ when $-1+1/p < s < 1/p$. The proof proceeds by changing variables to flatten the boundary. As the discussion on p. 172 of Triebel indicates, when $p > 1$, the needed results for Lipschitz change of variables can be proven for $s = 0, 1$ using the chain rule (for $s = 1$) and change of variables. Then, we interpolate to obtain results for $0 < s < 1$. ∎

The next proposition uses the equality of the Dirichlet to Neumann maps to deduce the equality of expressions involving the solutions v_j constructed above. In this proposition and below, we will use a Schrödinger operator with a potential q_j which is defined by $\Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$. Since, in general, the coefficient γ_j does not have two derivatives, we define the Laplacian in a weak sense. To be precise, we will define the pairing between q_j and a function ϕ with one derivative by

$$\langle q_j, \phi \rangle = -\int_{\mathbf{R}^n} \nabla\sqrt{\gamma_j} \cdot \nabla \frac{\phi}{\sqrt{\gamma_j}}\, dx. \tag{14}$$

Proposition 9 *Suppose $n \geq 3$. Let γ_1 and γ_2 satisfy (13) and suppose that $\nabla\sqrt{\gamma_j}$ is in L^n_c. Suppose further that $\gamma_1 = \gamma_2$ outside Ω and that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. We let $q_j = \Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$. If for $j = 1, 2$, v_j is a solution of $\Delta v_j - q_j v_j = 0$ which lies in $W^{1,2}_{loc}(\mathbf{R}^n)$, then*

$$\langle q_1, v_1 v_2 \rangle = \langle q_2, v_1 v_2 \rangle.$$

Proof. We first observe that the multiplication operator given by $\Delta\sqrt{\gamma_j}/\sqrt{\gamma}_j$ maps $W^{1,2}(\mathbf{R}^n)$ to $W^{-1,2}(\mathbf{R}^n)$. To see this, we use the definition of the distribution $\Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$ in (14) to obtain

$$\langle \frac{\Delta\sqrt{\gamma_j}}{\sqrt{\gamma_j}}, uv \rangle = -\int_{\mathbf{R}^n} \nabla\sqrt{\gamma_j} \cdot \nabla \frac{uv}{\sqrt{\gamma}_j}\, dx. \tag{15}$$

The product rule, Hölder's inequality, and then the Sobolev inequality (which requires $n \geq 3$), imply that

$$\left| \int \nabla\sqrt{\gamma_j} \cdot \nabla \frac{uv}{\sqrt{\gamma_j}}\, dx \right| \leq \|u\|_{W^{1,2}B_{R_0}} \|v\|_{W^{1,2}B_{R_0}} (\|\nabla \log \sqrt{\gamma_j}\|_{L^n} + \|\nabla \log \sqrt{\gamma_j}\|^2_{L^n}).$$

Thus the expressions in (15) are defined, at least when $n \geq 3$.

Now, we turn to the proof of the equality of the theorem. First, we consider the integral outside of Ω. Since we have $\sqrt{\gamma_1} = \sqrt{\gamma_2}$ outside Ω, it follows immediately that

$$\int_{\mathbf{R}^n \setminus \Omega} \nabla \sqrt{\gamma_1} \cdot \nabla \frac{v_1 v_2}{\sqrt{\gamma_1}} \, dx = \int_{\mathbf{R}^n \setminus \Omega} \nabla \sqrt{\gamma_2} \cdot \nabla \frac{v_1 v_2}{\sqrt{\gamma_2}} \, dx. \tag{16}$$

To study the integral inside Ω, we observe that if we define $u_j = v_j / \sqrt{\gamma_j}$, then we have that $\mathrm{div} \gamma_j \nabla u_j = 0$. This follows from a well-known calculation which we omit. Next, we claim that

$$\int_\Omega \nabla \sqrt{\gamma}_1 \cdot \nabla (\frac{1}{\sqrt{\gamma_1}} v_1 v_2) - \nabla v_1 \cdot \nabla v_2 \, dx = -\Lambda_{\gamma_1}(u_1)(u_2). \tag{17}$$

Now, if we interchange the indices 1 and 2 in (17), subtract the result from (17), use that $\Lambda_{\gamma_j}(f)(g) = \Lambda_{\gamma_j}(g)(f)$, and then our assumption that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ we obtain that

$$\int_\Omega \nabla \sqrt{\gamma}_1 \cdot \nabla \frac{v_1 v_2}{\sqrt{\gamma_1}} \, dx = \int_\Omega \nabla \sqrt{\gamma_2} \cdot \nabla \frac{v_1 v_2}{\sqrt{\gamma_2}} \, dx. \tag{18}$$

If we add (16) and (18), we obtain the conclusion of the theorem.

Thus, we turn to the proof of (17) and for this we will need an additional function $\tilde{u}_2 = v_2 / \sqrt{\gamma_1}$. By the definitions of u_1 and $\tilde{u}_2$ and the product rule, we obtain

$$\begin{aligned} &\int_\Omega \nabla \sqrt{\gamma_1} \cdot \nabla (\frac{1}{\sqrt{\gamma_1}} v_1 v_2) - \nabla v_1 \cdot \nabla v_2 \, dx \\ &\quad = \int_\Omega \nabla \sqrt{\gamma_1} \cdot \nabla (\sqrt{\gamma_1} u_1 \tilde{u}_2) - u_1 \nabla \sqrt{\gamma_1} \cdot \nabla (\sqrt{\gamma_1} \tilde{u}_2) \\ &\qquad\quad - \sqrt{\gamma_1} \nabla u_1 \cdot (\tilde{u}_2 \nabla \sqrt{\gamma_1}) - \gamma_1 \nabla u_1 \cdot \nabla \tilde{u}_2 \, dx \\ &\quad = -\Lambda_{\gamma_1}(u_1)(u_2). \end{aligned}$$

In the last equality, we use that $u_2 - \tilde{u}_2$ is in $W^{1,2}_0(\Omega)$ and hence they restrict to the same element in $W^{1/2,2}(\partial\Omega)$. This is a consequence of Lemma 10 below. This completes the proof of (17) and hence the Proposition. ∎

Lemma 10 *Assume $\partial\Omega$ is Lipschitz and that Ω is bounded. If β is $C(\bar{\Omega}) \cap W^{1,n}(\Omega)$ and $\beta(x) = 0$ for $x \in \partial\Omega$, then the map $u \to \beta u$ maps $W^{1,2}(\Omega)$ to $W^{1,2}_0(\Omega)$.*

Proof. We let $\epsilon > 0$ and construct a cutoff function $\eta_\epsilon(x)$ where $\eta_\epsilon(x) = 1$ if $\delta(x) > 2\epsilon$ and $\eta_\epsilon(x) = 0$ if $\delta(x) < \epsilon$. Here, we are using $\delta(x)$ to denote the distance from x to $\partial\Omega$. This function may be constructed to satisfy $|\nabla \eta_\epsilon| \leq C/\epsilon$. We consider $\nabla(\eta_\epsilon \beta) - \nabla \beta = (\eta_\epsilon - 1)\nabla\beta + \beta \nabla(\eta_\epsilon - 1)$. By the dominated convergence theorem, we have $\lim_{\epsilon \to 0^+} \int_\Omega |\nabla \beta|^n |(1 - \eta_\epsilon)|^n = 0$. While Hardy's inequality (see [7, p. 28], for example) implies that

$$\int_\Omega |\beta|^n |\nabla \eta_\epsilon|^n \, dx \leq \int_{\{x : \delta(x) < C\epsilon\}} |\nabla \beta|^n \, dx.$$

Hence, we can approximate β in the norm of $W^{1,n}(\Omega)$ by $\beta\eta_\epsilon$. Furthermore, by regularizing, we can find a sequence of functions $\{\beta_j\}$ which are infinitely differentiable, converge to β in $W_0^{1,n}(\Omega)$, the sequence is bounded in $L^\infty(\Omega)$ and converges pointwise to β. Now, we claim that $\beta_j u$ converges to βu in $W^{1,2}(\Omega)$. The convergence of $\beta_j u$ to βu in L^2 follows from the dominated convergence theorem. Now consider $\nabla(\beta_j u) = u\nabla\beta_j + \beta_j \nabla u$. The convergence of $\beta_j \nabla u$ to $\beta\nabla u$ follows from the dominated convergence theorem. We have

$$\|u(\nabla\beta_j - \nabla\beta)\|_{L^2(\Omega)} \leq \|\nabla\beta_j - \nabla\beta\|_{L^n(\Omega)}\|u\|_{L^{2n/(n-2)}(\Omega)}. \tag{19}$$

From the inequality (19), we see that the convergence of $\nabla\beta_j$ in L^n implies the convergence of $u\nabla\beta_j$ to $u\nabla\beta$ in L^2. We have established that $\|\nabla(\beta_j u) - \nabla(\beta u)\|_{L^n(\Omega)}$ goes to zero as $j \to \infty$. The term $\|u\|_{L^{2n/(n-2)}(\Omega)}$ is finite because of the Sobolev embedding theorem. Hence, the right-hand side of (19) tends to zero. Because the sequence of functions $\beta_j u$ tend to βu in $W^{1,2}(\Omega)$ we can conclude that βu is in $W_0^{1,2}(\Omega)$.
∎

Finally, we give the proof of our main theorem. Given the above, the proof of the main result follows familiar lines.

Theorem 11 *Suppose $n \geq 3$, $\partial\Omega$ is Lipschitz, $\gamma_j \in W^{3/2,2n+\epsilon}(\Omega)$ for some $\epsilon > 0$ and that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then we have that $\gamma_1 = \gamma_2$.*

Proof. Because of the equality of the Dirichlet to Neumann maps, we may use the result of Corollary 8 to extend γ_1 and γ_2 to all of $\mathbf{R}^n$ so that $\gamma_1 = \gamma_2$ in $\mathbf{R}^n \setminus \Omega$. Thus, according to Proposition 9 we have

$$\langle (q_1 - q_2), v_1 v_2 \rangle = 0 \tag{20}$$

where q_j are the potentials, $\Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$ as above and v_j are solutions of $\Delta v_j - q_j v_j = 0$. Now, we fix $\xi \in \mathbf{R}^n$, let $R > 0$ be large and construct ζ_1 and ζ_2 in $\mathbf{C}^n$ satisfying

$$\zeta_j \cdot \zeta_j = 0, \qquad j = 1, 2 \tag{21}$$

$$\zeta_1 + \zeta_2 = -i\xi, \qquad j = 1, 2 \tag{22}$$

$$|\zeta_j| > R, \qquad j = 1, 2. \tag{23}$$

We recall the standard construction of ζ_j. Choose e_1 and e_2 unit vectors in $\mathbf{R}^n$ and so that e_1, e_2 and ξ are mutually orthogonal and then put $\zeta_1 = -Re_1 - ie_2\sqrt{R^2 - |\xi|^2/4} - i\xi/2$ and $\zeta_2 = Re_1 + ie_2\sqrt{R^2 - |\xi|^2/4} - i\xi/2$. We construct the solutions v_j of $\Delta v_j - q_j v_j = 0$ corresponding to ζ_j as given by Corollary 6. Note that the solutions given by this Corollary must have $3/2$ derivatives L^2_{loc} and thus they satisfy the hypotheses of Proposition 9. To see this, observe that the right-hand side of $\Delta v = qv$ lies $W^{-1/2,2}_{loc}$

and thus by elliptic regularity, v has two more derivatives. We substitute $e^{x\cdot\zeta_j}(1+\psi_j)$ for v_j in (20) and obtain

$$0 = \langle q_1 - q_2, e^{-ix\cdot\xi}(1+\psi_1+\psi_2+\psi_1\psi_2)\rangle.$$

Because the functions ψ_j tend to zero in $B^*_{1/2}$ as $R \to \infty$, passing to the limit in the previous equation gives $\hat{q}_1 = \hat{q}_2$. Here, we must use Theorem 1 and our estimate for ψ_1, ψ_2 in $B^*_{1/2}$ to conclude that the product $\psi_1\psi_2$ goes to zero in $W^{1/2,2n/(2n-1)}_{loc}$. Hence, it follows that $\Delta\sqrt{\gamma_1}/\sqrt{\gamma_1} = \Delta\sqrt{\gamma_2}/\sqrt{\gamma_2}$. As in [3, p. 1056], this implies that $\log(\gamma_1/\gamma_2)$ solves $\mathrm{div}\sqrt{\gamma_1\gamma_2}\nabla\log(\gamma_1/\gamma_2) = 0$ and is compactly supported and hence vanishes by the weak maximum principle. ∎

We close with several questions motivated by the above work.

1. Can we obtain uniqueness in the inverse conductivity problem at the endpoint $p = 2n$? The above argument requires $p > 2n$ in order to carry out the extension in Corollary 8.

2. Can we lower the L^p-space from $p = 2n$ to $p = 2n/3$ in the uniqueness result for conductivities with 3/2 derivatives?

3. Can we lower the smoothness index to 1 and obtain a uniqueness result?

4. The result for uniqueness at the boundary in Proposition 7 that we quote from Alessandrini [1] does not seem sharp. There is a boundary uniqueness result for the gradient of the conductivity which requires that the coefficient be continuously differentiable in the work of Sylvester and Uhlmann [17], however the domain must be smooth. The work of Brown [5] gives a way of reconstructing the boundary values for continuous (and some discontinuous) conductivities in Lipschitz domains. However, this work does not discuss the gradient of the conductivity. The work Tanuma and Nakamura [13] and Kang and Yun [11] gives boundary identifiability of the gradient of a $C^{1+\epsilon}$ conductivity in a $C^{2+\epsilon}$ domain. Results of Nachman [12] give boundary identifiability for the gradient of $C^{1,1}$ conductivities in domains with $C^{1,1}$ boundaries. A reasonable conjecture is that we can determine the gradient of continuously differentiable conductivities at the boundary in a Lipschitz domain. This result does not seem to be in the literature.

References

[1] G. Alessandrini. Singular solutions of elliptic equations and the determination of conductivity by boundary measurements. *J. Diff. Equations*, 84:252–272, 1990.

[2] B. Barceló, C. E. Kenig, A. Ruiz, and C. D. Sogge. Weighted Sobolev inequalities and unique continuation for the Laplacian plus lower order terms. *Illinois J. Math.*, 32(2):230–245, 1988.

[3] R.M. Brown. Global uniqueness in the impedance imaging problem for less regular conductivities. *SIAM J. Math. Anal.*, 27:1049–1056, 1996.

[4] R.M. Brown and G.A. Uhlmann. Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions. *Comm. Partial Differential Equations*, 22(5-6):1009–1027, 1997.

[5] Russell M. Brown. Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result. To appear, J. Inv. Ill-posed problems.

[6] A. P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and its Applications to Continuum Physics*, pages 65–73, Rio de Janeiro, 1980. Soc. Brasileira de Matemática.

[7] P. Grisvard. *Elliptic problems in nonsmooth domains*. Pitman, 1985.

[8] G.M. Henkin and R.G. Novikov. A multidimensional inverse problem in quantum and acoustic scattering. *Inverse problems*, 4:103–121, 1988.

[9] L. Hörmander. *The analysis of linear partial differential operators. I*. Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.

[10] L. Hörmander. *The analysis of linear partial differential operators. II*. Springer-Verlag, Berlin, 1983. Differential operators with constant coefficients.

[11] H. Kang and K.H. Yun. Boundary determination of conductivities and riemannian metrics via local dirichlet-to-neumann operator. Preprint, 2001.

[12] A.I. Nachman. Reconstructions from boundary measurements. *Annals of Math.*, 128:531–587, 1988.

[13] G. Nakamura and K. Tanuma. Local determination of conductivity at the boundary from the Dirichlet-to-Neumann map. *Inverse Problems*, 17(3):405–419, 2001.

[14] L. Päivärinta, A. Panchenko, and G.A. Uhlmann. Complex geometrical optics solutions for Lipschitz conductivities. To appear, *Rev. Mat. Ibero.*, 2000.

[15] T. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*. Walter de Gruyter & Co., Berlin, 1996.

[16] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Annals of Math.*, 125:153–169, 1987.

[17] J. Sylvester and G. Uhlmann. Inverse boundary value problems at the boundary–continuous dependence. *Comm. Pure Appl. Math.*, 61:197–219, 1988.

[18] H. Triebel. *Theory of function spaces*. Birkhäuser Verlag, Basel, 1983.

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