Estimates for the Stokes Operator in Lipschitz Domains

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Abstract: We study the Stokes operator A in a three-dimensional Lipschitz domain Ω. Our main result asserts that the domain of A is contained in $W_0^{1,p}(\Omega) \cap W^{3/2,2}(\Omega)$ for some $p > 3$. Certain L^∞-estimates are also established. Our results may be used to improve the regularity of strong solutions of Navier-Stokes equations in nonsmooth domains. In the appendix we provide a simple proof of area integral estimates for solutions of Stokes equations.

Introduction

In a recent interesting paper, Deuring and von Wahl [DW] consider strong solutions of the nonstationary Navier–Stokes equations in $\Omega \times (0,T)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - (u \cdot \nabla)u - \nabla \pi + f, \\ \operatorname{div} u = 0, \end{cases}$$

with the initial-Dirichlet condition

$$\begin{cases} u(X,t) = 0 \ \text{ for } \ (X,t) \in \partial\Omega \times (0,T), \\ u(X,0) = u_0(X) \ \ X \in \Omega, \end{cases}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^3$. Based on the functional analytical approach of Fujita and Kato [FK] and the Rellich estimates of Shen [S1], they show that, if $u_0 \in \mathcal{D}(A^{1/4+\varepsilon})$ for some $\varepsilon \in (0,1/2)$ and f is bounded and locally Hölder continuous, then a solution (u,π) exists for some $T > 0$ and

$$u \in C((0,T], \mathcal{D}(A)) \cap C([0,T], \mathcal{D}(A^{\frac14 + \varepsilon}))$$

where $A = -P\Delta$ denotes the Stokes operator.

The purpose of this note is to describe $\mathcal{D}(A)$, the domain of A, in terms of Sobolev's spaces. In the case of smooth domains, it is well known that

$$\mathcal{D}(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \cap L^2_\sigma(\Omega)$$

where $L^2_\sigma(\Omega)$ denotes the space of solenoidal functions in $L^2(\Omega)$ (e.g., see [CF]). One can not expect such results in Lipschitz domains, as the $W^{2,2}$–estimate in general fails in nonsmooth domains. Our main results in this paper assert that

$$(0.1) \qquad\qquad \mathcal{D}(A) \subset W_0^{1,p}(\Omega) \cap W^{3/2,2}(\Omega)$$

for some $p = p(\Omega) > 3$ (Theorem 2.17). In particular, it follows from the Sobolev's imbedding that for every $t \in (0,T]$, $u(t) \in C^\alpha(\overline{\Omega})$ for some $\alpha = \alpha(\Omega) > 0$, i.e.,

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the strong solution of the Navier-Stokes equations is Hölder continuous up to the boundary as a function of X. We also obtain the following L^∞ estimates:

$$(0.2) \qquad \begin{aligned} \|u\|_{L^\infty(\Omega)} &\leq C\|\nabla u\|_{L^2(\Omega)}^{1/2}\|Au\|_{L^2(\Omega)}^{1/2}, \\ \|u\|_{L^\infty(\Omega)} &\leq C\|u\|_{L^2(\Omega)}^{1/4}\|Au\|_{L^2(\Omega)}^{3/4} \end{aligned}$$

for $u \in \mathcal{D}(A)$. To establish (0.2), we use the reverse Hölder inequality, (0.1) and some localization techniques. See Theorem 3.1 and Corollary 3.2. Estimates like (0.2) are very useful in the study of Navier-Stokes equations. See [CF] and [H] in the case of smooth domains.

To prove (0.1), we shall study the Dirichlet problem for the Stokes equations with a forcing term, and interpolate between the L^2 estimates in [FKV] and the Hölder estimates in [S2]. The following area integral estimate:

$$\int_\Omega |\nabla u(X)|^2 \mathrm{dist}(X, \partial\Omega) dX \leq C \int_{\partial\Omega} |u|^2 d\sigma$$

for solutions of Stokes equations $\Delta u = \nabla\pi$, $\mathrm{div}\, u = 0$ in Ω, plays an important role. This estimate is due to E. Fabes, C. Kenig and G. Verchota, but no proof has appeared in the literature. In the appendix of this paper, which is due to Z. Shen, we will provide a simple proof of the area integral estimates for solutions of Stokes equations. The proof given here is based on the idea of a recent paper by B. Dahlberg, C. Kenig, J. Pipher, and G. Verchota [DKPV] for higher order equations and systems, together with some observations on the pressure term π.

Finally, the second author would like to thank C. Kenig for many helpful conversations.

1. Notation and Definitions

In this section we collect the definitions for Lipschitz domains, the nontangential maximal function, the Sobolev and Besov spaces we will use, and the Stokes operator. We shall also recall a few elementary facts regarding complex interpolation.

Lipschitz Domains. Let Ω be a bounded, open, connected set in $\mathbb{R}^n$. We say that Ω is a *Lipschitz domain* if for each $P \in \partial\Omega$, there exists a coordinate system (x', x_n), which is isometric to the usual coordinates, and a Lipschitz function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$, a radius $r > 0$ so that

$$B(P, 2r) \cap \Omega = \{(x', x_n) \in \mathbb{R}^n : x_n > \psi(x')\} \cap B(P, 2r),$$
$$B(P, 2r) \cap \partial\Omega = \{(x', x_n) \in \mathbb{R}^n : x_n = \psi(x')\} \cap B(P, 2r).$$

The Nontangential Maximal Function. For a function u on Ω, we define its *nontangential maximal function* $(u)^*$ by

$$(1.1) \qquad (u)^*(P) = \sup\{|u(X)| : X \in \Omega, \ |X - P| \leq 2\mathrm{dist}(X, \partial\Omega)\} \quad P \in \partial\Omega.$$

We now give the definition of the function spaces we will use.

Sobolev and Besov Spaces. For $\Omega \subset \mathbb{R}^n$, $p \in [1, \infty)$ and $k = 1, 2, \ldots$, we let $W^{k,p}(\Omega)$ denote the space of functions u on Ω such that the norm

$$\|u\|_{W^{k,p}(\Omega)} \equiv \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}$$

is finite.

For $1 \leq p < \infty$ and $0 < \alpha < 1$, we define $W^{\alpha,p}(\Omega)$ to be the collection of functions u on Ω with the norm

$$(1.2) \qquad \|u\|_{W^{\alpha,p}(\Omega)} \equiv \|u\|_{L^p(\Omega)} + \left(\iint\limits_{\Omega \times \Omega} \frac{|u(X) - u(Y)|^p}{|X - Y|^{n+\alpha p}} dX dY \right)^{1/p} < \infty.$$

For $1 < \alpha < 2$, we define $W^{\alpha,p}(\Omega)$ to be the set of functions on Ω such that

$$\|u\|_{W^{\alpha,p}(\Omega)} \equiv \|u\|_{L^p(\Omega)} + \|\nabla u\|_{W^{\alpha-1,p}(\Omega)} < \infty.$$

We may define $W^{\alpha,p}(\partial\Omega)$ for $0 < \alpha < 1$ and $1 \leq p < \infty$ in a similar manner with the integral over $\partial\Omega \times \partial\Omega$.

For a region D above the graph of a Lipschitz function ϕ, i.e., $D = \{(x', x_n) \in \mathbb{R}^n : x_n > \phi(x')\}$, we define the space $W^{1,2}(\partial D)$ as the space of functions $f(x', \phi(x')) = g(x')$ where $g \in W^{1,2}(\mathbb{R}^{n-1})$. Using a partion of unity for $\partial\Omega$, we may extend this definition to the boundary of a bounded Lipschitz domain Ω for $W^{1,2}(\partial\Omega)$. We remark that $W^{\alpha,p}(\partial\Omega)$ may also be defined in this manner. See [Gr, p. 20].

We will use $W_0^{k,p}(\Omega)$ to denote the closure of $C_0^\infty(\Omega)$ in the norm of $W^{k,p}(\Omega)$. If k is a nonnegative integer, we use $W^{k,p}(\Omega)$ to denote the dual of $W_0^{-k,p}(\Omega)$.

We will use the same notation $L^p(\Omega)$, $W^{\alpha,p}(\Omega)$, $W^{k,p}(\Omega)$ etc. for vectorial counterparts.

Complex Interpolation. We will need the following results on complex interpolation:

$$(1.3) \qquad \left[L^2(\partial\Omega), W^{1,2}(\partial\Omega) \right]_\theta = W^{\theta,2}(\partial\Omega),$$

and

$$(1.4) \qquad \left[W^{\alpha,2}(\partial\Omega), C^\gamma(\partial\Omega) \right]_\theta = W^{t,p}(\partial\Omega)$$

where

$$\frac{1}{p} = \frac{\theta}{2}, \quad \text{and} \quad t = \alpha\theta + \gamma(1 - \theta).$$

When $\partial\Omega$ is replaced by $\mathbb{R}^n$, (1.3) and (1.4) are well known (e.g. see [BL]). To extend this result to boundaries of Lipschitz domains, we use the following easy proposition, whose proof is omitted.

Proposition 1.5. *Suppose A_i, B_i $(i = 0, 1)$ are Banach spaces and $\{C(\theta) : \theta \in [0, 1]\}$ is a family of Banach spaces such that $C(0) = A_0$, $C(1) = A_1$ and $C(\theta) \subset A_0 + A_1$. Also assume that there exist linear operators $I : A_0 + A_1 \to B_0 + B_1$ and $P : B_0 + B_1 \to A_0 + A_1$ such that $I : C(\theta) \to [A_0, A_1]_\theta$, $P : [A_0, A_1]_\theta \to C(\theta)$ are bounded and $P \circ I = id$ for each $\theta \in [0, 1]$. Then*

$$[A_0, A_1]_\theta = C(\theta).$$

To see (1.3), let $\{B_j = B(P_j, r) : j = 1, 2, \ldots, N\}$ be a covering of $\partial\Omega$ by balls as in the definition of Lipschitz domain and $\eta_j \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\eta_j \subset B_j$, $0 \le \eta_j \le 1$, $\sum_j \eta_j = 1$ on $\partial\Omega$. Let ϕ_j be an isometry of $\mathbb{R}^n$ such that

$$\phi_j(B_j \cap \Omega) = \{(x', x_n) : x_n > \psi_j(x'), \ |(x', x_n)| \le r\}.$$

We now choose $A_0 = L^2(\partial\Omega)$, $A_1 = W^{1,2}(\partial\Omega)$, $B_0 = L^2(\mathbb{R}^{(n-1)}, \mathbb{R}^N)$, $B_1 = W^{1,2}(\mathbb{R}^{(n-1)}, \mathbb{R}^N)$, and $C(\theta) = W^{\theta,2}(\partial\Omega)$. If g is a function of $\partial\Omega$, we define Ig by letting the jth component be given by

$$Ig(x')_j = (\eta_j g)(\phi_j^{-1}(x', \psi_j(x'))).$$

We let $\tilde\eta_j \in C_0^\infty(B(P_j, 2r))$ $j = 1, \ldots, N$ be functions satisfying $\tilde\eta_j = 1$ on $B(P_j, r)$, we let $\pi_n(x', x_n) = x'$ be projection on the first $n - 1$ coordinates and define

$$P((f_j)_{1 \le j \le N})(Q) = \sum_{j=1}^N \tilde\eta_j(Q) f_j(\pi_n \circ \phi_j(Q)) \quad \text{for} \ Q \in \partial\Omega.$$

With these definitions, (1.3) then follows easily from Proposition 1.5. The statement (1.4) may be proved in the same manner.

The Stokes Operator. To introduce the Stokes operator A, let

$$C_{0,\sigma}^\infty(\Omega) = \{\phi \in C_0^\infty(\Omega) : \operatorname{div} \phi = 0\},$$

and $L_\sigma^2(\Omega)$ be the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)$. We first define a quadratic form Q on $C_{0,\sigma}^\infty(\Omega)$:

$$Q(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dX = \sum_{j=1}^n \int_\Omega \nabla u_j \cdot \nabla v_j \, dX.$$

We then extend this form to $\mathcal{D}(Q)$, the closure of $C_{0,\sigma}^\infty(\Omega)$ in the norm

$$\|u\|_Q = \|u\|_{L^2(\Omega)} + \{Q(u, u)\}^{1/2}.$$

It is known that

$$\mathcal{D}(Q) = \left\{u \in W_0^{1,2}(\Omega) : \operatorname{div} u = 0\right\}$$

[CF].

We now define the Stokes operator $A : \mathcal{D}(A) \to L_\sigma^2(\Omega)$ by

$$\int_\Omega Au \cdot \phi = Q(u, \phi) \ \text{ for all } \ \phi \in C_{0,\sigma}^\infty(\Omega)$$

where the domain of A, $\mathcal{D}(A)$, is the collection of u in $\mathcal{D}(Q)$ such that $v \to Q(u, v)$ can be extended to a bounded linear functional on $L_\sigma^2(\Omega)$. It is well known that A gives a self-adjoint operator on $L_\sigma^2(\Omega)$. Also, it is not hard to see that

$$\mathcal{D}(A) = \left\{u \in \mathcal{D}(Q) : -\Delta u + \nabla\pi \in L_\sigma^2(\Omega) \text{ for some } \pi \in L^2(\Omega)\right\}$$

and
$$Au = -\Delta u + \nabla \pi, \qquad \text{for} \ \ u \in \mathcal{D}(A).$$

This is the definition of the Stokes operator given in [DW].

Finally we will follow the standard practice of letting C denote a constant which varies. Throughout this paper C will depend at most in the dimension n and the Lipschitz domain Ω through the collection of balls used to cover the boundary and the maximum of $\|\nabla \psi_i\|_\infty$ where ψ_i are the functions whose graphs define $\partial \Omega$.

2. The Imbedding of $\mathcal{D}(A)$

In this section we will establish the imbedding estimate (0.1) for the domain of A.

We will need to consider both the Dirichlet problem with nonzero forcing term f

(DP)
$$\begin{cases} -\Delta u + \nabla \pi = f & \text{in} \ \ \Omega, \\ \operatorname{div} u = 0 & \text{in} \ \ \Omega, \\ u = 0 & \text{on} \ \ \partial \Omega, \end{cases}$$

and the problem with inhomogeneous boundary data:

(BVP)
$$\begin{cases} -\Delta u + \nabla \pi = 0 & \text{in} \ \ \Omega, \\ \operatorname{div} u = 0 & \text{in} \ \ \Omega, \\ u = g & \text{on} \ \ \partial \Omega. \end{cases}$$

Obviously, since $\operatorname{div} u = 0$ in Ω, the boundary data g should verify the necessary condition

(2.1)
$$\int_{\partial \Omega} < g, N > d\sigma = 0$$

where N denotes the outward unit normal to $\partial \Omega$ and $<,>$ the scalar product on $\mathbb{R}^n$.

The following result is due to Fabes, Kenig and Verchota [FKV]. The estimates for the nontangential maximal functions, the existence and the uniqueness may be found in their paper. The estimates for the solution in Sobolev spaces were announced in that paper, but no proof has appeared. The proof may be obtained by combining the area integral estimates in the appendix of this paper, with the argument given by E. Fabes in [F] to establish the corresponding Sobolev estimates for harmonic functions. We remark that in the first inequality on the top of page 69 of [F], the integration on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ should only range over $\{(x,y) : |x - y| > t\}$.

Theorem 2.2. *(Fabes, Kenig and Verchota* [FKV]*) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. Suppose $g \in L^2(\partial \Omega)$ and verifies the condition (2.1). Then there exists a solution u to (BVP) which satisfies*

$$\int_{\partial \Omega} |(u)^*|^2 d\sigma \leq C \int_{\partial \Omega} |g|^2 d\sigma.$$

This solution is unique in the class of u satisfying $(u)^* \in L^2(\partial\Omega)$. *In addition, we have the estimate*

$$\|u\|_{W^{1/2,2}(\Omega)} \leq C \int_{\partial\Omega} |g|^2 d\sigma.$$

Furthermore, if $g \in W^{1,2}(\partial\Omega)$, *then we also have*

$$\|(\nabla u)^*\|_{L^2(\partial\Omega)} + \|u\|_{W^{3/2,2}(\Omega)} \leq C\|g\|_{W^{1,2}(\partial\Omega)}.$$

We shall also need the Hölder estimates established in [S2, Theorem 0.2, p. 801] for the three-dimensional Lipschitz domains.

Theorem 2.3. *(Z. Shen [S2]) Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^3$. *There exists* $\alpha_0 > 0$ *so that if* $0 < \alpha < \alpha_0$, *and* $g \in C^\alpha(\partial\Omega)$ *verifies the condition (2.1), then the solution to (DP) lies in* $C^\alpha(\overline{\Omega})$ *and satisfies*

$$\sup_{X \in \Omega} \delta(X)^{1-\alpha} |\nabla u(X)| \leq C\|g\|_{C^\alpha(\partial\Omega)}$$

where $\delta(X) = dist(X, \partial\Omega)$.

Our estimates for the Stokes operator A will be obtained by interpolating between the estimates of Theorem 2.2 and 2.3. We begin with the following lemma.

Lemma 2.4. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^3$. *There exists* $\varepsilon > 0$ *so that if* $2 \leq p \leq 3 + \varepsilon$, *and* $g \in W^{1-1/p,p}(\partial\Omega)$ *satisfies the condition (2.1), then the solution* u *of (BVP) for Stokes equations with boundary data* g *satisfies*

$$\left(\int_\Omega |\nabla u(X)|^p dX \right)^{1/p} \leq C\|g\|_{W^{1-1/p,p}(\partial\Omega)}.$$

Proof. Let η be a smooth vector field on $\mathbb{R}^3$ such that $<\eta, N> \geq c_0 > 0$ a.e. on $\partial\Omega$. Recall that N is the outward unit normal to $\partial\Omega$.

For $g \in W^{\alpha,p}(\partial\Omega)$, we define Sg by

$$Sg(P) = g(P) - \frac{\int_{\partial\Omega} <g, N> d\sigma}{\int_{\partial\Omega} <\eta, N> d\sigma} \cdot \eta(P) \quad \text{for} \quad P \in \partial\Omega.$$

It is easy to see that

$$\int_{\partial\Omega} <Sg, N> d\sigma = 0.$$

Now let u be the solution of (BVP) with boundary data Sg. We observe that Theorem 2.2, the area integral estimate in Appendix A (Theorem A.1) and interior estimates imply that

$$\left(\int_\Omega \delta(X)^3 |\nabla^2 u(X)|^2 dX \right)^{1/2} \leq C\|Sg\|_{L^2(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)},$$

$$\left(\int_\Omega \delta(X) |\nabla^2 u(X)|^2 dX \right)^{1/2} \leq C\|Sg\|_{W^{1,2}(\partial\Omega)} \leq C\|g\|_{W^{1,2}(\partial\Omega)}.$$

It then follows from the complex interpolation that

$$(2.5) \qquad \left(\int_\Omega \left(\delta(X)^{\frac{3}{2}-\alpha} |\nabla^2 u(X)| \right)^2 dX \right)^{1/2} \leq C \|g\|_{W^{\alpha,2}(\partial\Omega)}, \quad 0 < \alpha < 1.$$

where we have used (1.3).

Now, since $\Omega \subset \mathbb{R}^3$, we may apply Theorem 2.3 and the interior estimates to obtain

$$(2.6) \qquad \sup_{X \in \Omega} \left\{ \delta(X)^{2-\gamma} |\nabla^2 u(X)| \right\} \leq C \|Sg\|_{C^\gamma(\partial\Omega)} \leq C \|g\|_{C^\gamma(\partial\Omega)}$$

for $0 < \gamma < \alpha_0$. If we interpolate between (2.5) and (2.6) and use (1.4), we obtain that

$$(2.7) \qquad \left(\int_\Omega \left\{ \delta(X) |\nabla^2 u(X)| \right\}^p dX \right)^{1/p} \leq C \|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}$$

for $2 < p < 3 + \varepsilon$.

To see this, set $\theta = 2/p < 1$, and choose α and γ so that

$$1 = (\frac{3}{2} - \alpha)\theta + (2 - \gamma)(1 - \theta).$$

Thus,

$$t = \alpha\theta + \gamma(1 - \theta) = 1 - \frac{1}{p}.$$

We use (1.4) to identify the interpolation space

$$\left[W^{\alpha,2}(\partial\Omega), C^\gamma(\partial\Omega) \right]_\theta = W^{t,p}(\partial\Omega).$$

Note that $2/p < 1$ implies that $p > 2$ and the restrictions that $\alpha < 1$ and $\gamma < \alpha_0$ imply that

$$\theta = \frac{1-\gamma}{\alpha + \frac{1}{2} - \gamma} > \frac{2(1-\alpha_0)}{3 - 2\alpha_0}$$

or

$$p = \frac{2}{\theta} < \frac{3 - 2\alpha_0}{1 - \alpha_0} \equiv 3 + \varepsilon, \quad \varepsilon = \frac{\alpha_0}{1 - \alpha_0} > 0.$$

Finally, by the Hardy inequality [St, p. 272],

$$\int_\Omega |\nabla u(X)|^p dX \leq C \int_\Omega |\delta(X)\nabla^2 u(X)|^p dX + \sup_{X \in K} |\nabla u(X)|^p$$

$$\leq C \|g\|^p_{W^{1-\frac{1}{p},p}(\partial\Omega)},$$

where K is a compact subset of Ω. The proof is complete.

Let $\Gamma(X) = (\Gamma_{ij}(X))_{1 \leq i,j \leq n}$ be the matrix of fundamental solutions and $q(X) = (q^j(X))_{1 \leq j \leq n}$ be the corresponding pressure vector where

$$(2.8) \qquad \Gamma_{ij}(X) = \frac{1}{2\omega_n} \left(\frac{1}{(n-2)} \frac{\delta_{ij}}{|X|^{n-2}} + \frac{x_i x_j}{|X|^n} \right), \qquad q^j(X) = \frac{1}{\omega_n} \frac{x_j}{|X|^n}.$$

Theorem 2.9. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Suppose $f \in W^{-1,p}(\Omega)$ where $(3+\varepsilon)/(2+\varepsilon) < p < 3+\varepsilon$ and ε is the same as in Lemma 2.4. Then there exist $u \in W_0^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)$ so that*

$$(2.10) \qquad \begin{cases} -\Delta u + \nabla \pi = f, \\ div\ u = 0 \end{cases}$$

in Ω and

$$\|\nabla u\|_{L^p(\Omega)} + \inf_{c \in \mathbb{R}} \|\pi - c\|_{L^p(\Omega)} \le C\|f\|_{W^{-1,p}(\Omega)}.$$

The solution u is unique and π is unique up to a constant.

Proof. Let $2 < p < 3+\varepsilon$ and $f \in W^{-1,p}(\Omega)$. We may extend f to lie in $W^{-1,p}(\mathbb{R}^3)$. Let $v = \Gamma * f \in W^{1,p}(\mathbb{R}^3)$. By the trace theorem [Gr, p. 33], $v \in W^{1-1/p,p}(\partial\Omega)$ and we obtain

$$(2.11) \qquad \|v\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \le C\|v\|_{W^{1,p}(\Omega)} \le C\|f\|_{W^{-1,p}(\Omega)}.$$

Now let $u = v - w$ where w is the solution of (BVP) for the Stokes equations with boundary data $v|_{\partial\Omega}$. Then u satisfies (2.10) and

$$\|\nabla u\|_{L^p(\Omega)} \le \|\nabla v\|_{L^p(\Omega)} + \|\nabla w\|_{L^p(\Omega)}$$

$$\le C\left\{ \|f\|_{W^{-1,p}(\Omega)} + \|v\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \right\}$$

$$\le C\|f\|_{W^{-1,p}(\Omega)}$$

by Lemma 2.4 and (2.11).

We may obtain the existence of a solution and the estimates of u for $(3+\varepsilon)/(2+\varepsilon) < p < 2$ by duality.

For the pressure term π, we have

$$\inf_{c \in \mathbb{R}} \|\pi - c\|_{L^p(\Omega)} \le \|\nabla \pi\|_{W^{-1,p}(\Omega)} \le \|\Delta u\|_{W^{-1,p}(\Omega)} + \|f\|_{W^{-1,p}(\Omega)}$$

$$\le C\|\nabla u\|_{L^p(\Omega)} + \|f\|_{W^{-1,p}(\Omega)} \le C\|f\|_{W^{-1,p}(\Omega)}.$$

Finally, the uniqueness for $p \ge 2$ follows by energy estimates. If $p < 2$, let $-\Delta u + \nabla \pi = 0$ and $u \in W_0^{1,p}(\Omega)$. Choose $f \in L^{p'}(\Omega)$. We may solve $-\Delta v + \nabla q = f$, $div\ v = 0$ in Ω, $v \in W^{1,p'}(\Omega)$, and thus obtain that

$$\int_\Omega <f, u> dX = \int_\Omega \nabla v \cdot \nabla u\ dX = 0, \quad \text{for any } f \in L^{p'}(\Omega),$$

or that $u = 0$.

Our next result, which is valid for Lipschitz domains in $\mathbb{R}^n$ ($n \ge 3$), gives a sharper estimate for solutions of the Dirichlet problem. It will yield the best embedding of $\mathcal{D}(A)$ in the scale of Sobolev spaces $W^{s,2}(\Omega)$.

Theorem 2.12. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ ($n \geq 3$) and $f \in L^2(\Omega)$. Suppose that (u, π) is the solution of (DP) with data f, then for $q_0 = 2n/(n+1)$, we have the estimate*

$$\|u\|_{W^{3/2,2}(\Omega)} + \inf_{c \in \mathbb{R}} \|\pi - c\|_{W^{1/2,2}(\Omega)} \leq C\|f\|_{L^{q_0}(\Omega)}.$$

Proof. We construct the solution of (DP) as the sum of the free space solution $(v, \tilde{\pi}) = (\Gamma * f, q * f)$ and the solution of the boundary value problem

$$\begin{cases} -\Delta w + \nabla \psi = 0 & \text{in } \Omega, \\ \text{div } w = 0 & \text{in } \Omega, \\ w = -v & \text{on } \partial\Omega. \end{cases}$$

By the Calderón–Zygmund estimates [St], we have

$$(2.13) \qquad \|\nabla^2 v\|_{L^p(\mathbb{R}^n)} + \|\nabla \tilde{\pi}\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

for $1 < p < \infty$. This estimate with $p = q_0 = 2n/(n+1)$ and the Sobolev embedding theorem imply that

$$\|v\|_{W^{3/2,2}(\Omega)} + \|\tilde{\pi}\|_{W^{1/2,2}(\Omega)} \leq C\|f\|_{L^{q_0}(\Omega)}.$$

Now consider (w, ψ). We claim that the boundary values of w, $-\Gamma * f|_{\partial\Omega}$, satisfy

$$(2.14) \qquad \|w\|_{W^{1,2}(\partial\Omega)} \leq C\|f\|_{L^{q_0}(\Omega)}.$$

Then the desired estimate,

$$\|w\|_{W^{3/2,2}(\Omega)} + \|\psi\|_{W^{1/2,2}(\Omega)} \leq C\|f\|_{L^{q_0}(\Omega)}$$

will follow from Theorem 2.2.

To establish (2.14), we observe that

$$(2.15) \qquad \int_{\partial\Omega} |\nabla v|^2 d\sigma \leq C \int_{\Omega} \left(|\nabla v|^2 + |\nabla v| |\nabla^2 v| \right) dX.$$

This follows by applying the divergence theorem to $\eta \cdot |\nabla v|^2$ where η is a smooth vector field on $\mathbb{R}^n$ with the property $< \eta, N >\geq c_0 > 0$ a.e. on $\partial\Omega$. By (2.13), we have $|\nabla^2 v| \in L^{q_0}(\mathbb{R}^n)$ while

$$|\nabla v(X)| \leq C \int_{\Omega} \frac{|f(Y)|}{|X - Y|^{n-1}} dY$$

satisfies

$$(2.16) \qquad \|\nabla v\|_{L^{q_0'}(\mathbb{R}^n)} \leq C\|f\|_{L^{q_0}(\mathbb{R}^n)}, \quad q_0' = \frac{q_0}{q_0 - 1}.$$

Since $q_0' > 2$, the estimate for ∇w on $\partial\Omega$ follows from (2.15), (2.16) and (2.13). The estimate for w is easier and we omit the details.

Theorem 2.17. *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^3$. Then*

$$\mathcal{D}(A) \subset W_0^{1,p}(\Omega) \cap W^{3/2,2}(\Omega)$$

for some $p = p(\Omega) > 3$. Moreover, for $u \in \mathcal{D}(A)$,

$$\|u\|_{W_0^{1,p}(\Omega)} + \|u\|_{W^{3/2,2}(\Omega)} \le C\|Au\|_{L^2(\Omega)}.$$

Proof. Let $u \in \mathcal{D}(A)$. Then

$$\begin{cases} -\Delta u + \nabla \pi = f & \text{in } \Omega, \\ \text{div } u = 0 & \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

where $f = Au \in L_\sigma^2(\Omega)$. Since

$$L^2(\Omega) \subset W^{-1,p}(\Omega)$$

for $p \le 6$ in $\mathbb{R}^3$, the theorem follows immediately from Theorems 2.9 and 2.12.

3. Some L^∞ estimates

In this section we will give the proof of the estimate (0.2). We are only able to prove this estimate in three dimensions and thus throughout this section we will assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$.

Theorem 3.1. *There exists a constant $C > 0$ such that, for $u \in \mathcal{D}(A)$,*

$$\|u\|_{L^\infty(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}^{1/2}\|Au\|_{L^2(\Omega)}^{1/2}.$$

By the definition of A and a limiting argument,

$$\int_\Omega Au \cdot u \, dX = \int_\Omega \nabla u \cdot \nabla u \, dX.$$

It then follows from the Cauchy inequality that

$$\|\nabla u\|_{L^2(\Omega)}^2 \le \|Au\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.$$

Thus, we have

Corollary 3.2. *There exists a constant $C > 0$ such that, for $u \in \mathcal{D}(A)$,*

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{L^2(\Omega)}^{1/4}\|Au\|_{L^2(\Omega)}^{3/4}.$$

As in the case of smooth domains, $A^{-1} : L_\sigma^2(\Omega) \to L_\sigma^2(\Omega)$ is a compact operator. Hence, there exists a sequence of positive numbers $\{\lambda_k\}$ and an orthonormal basis $\{\omega_k\}$ of $L_\sigma^2(\Omega)$ such that $A\omega_k = \lambda_k \omega_k$, $\omega_k \in \mathcal{D}(A)$ and $\lim_{k\to\infty} \lambda_k = \infty$.

Corollary 3.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Then there exists $c > 0$ such that*

$$\lambda_k \geq c \left(\frac{k}{|\Omega|} \right)^{2/3}.$$

With Corollary 3.2, the proof is exactly the same as in the smooth case. See [CF, p. 38-39].

To prove Theorem 3.1, we start with a reverse Hölder inequality.

Lemma 3.4. *Let $X_0 \in \overline{\Omega}$, $R > 0$ be small, and $D(X_0, R) = B(X_0, R) \cap \Omega$. Assume (u, π) satisfies the Stokes equations*

$$\begin{cases} -\Delta u + \nabla \pi = 0, \\ div\, u = 0 \end{cases}$$

in $D(X_0, 8R)$ and $u = 0$ on $\partial\Omega$. Then

$$\left(\frac{1}{R^3} \int_{D(X_0, R)} |\nabla u|^{p_0} dX \right)^{1/p_0} \leq C \left(\frac{1}{R^3} \int_{D(X_0, 4R)} |\nabla u|^2 dX \right)^{1/2}$$

where $p_0 > 2$ depends only on Ω.

Proof. We begin with a Caccioppoli type inequality for the Stokes equations (see [S2, Lemma 1.5, p. 804]). We consider two cases. If $B(X_0, 2R) \cap \partial\Omega \neq \emptyset$, we have

$$\int_{D(X_0, R)} |\nabla u|^2 dX \leq \frac{C}{R^2} \int_{D(X_0, 3R)} |u|^2 dX.$$

On the other hand, if $B(X_0, 2R) \subset \Omega$, we get

$$\int_{D(X_0, R)} |\nabla u|^2 dX \leq \frac{C}{R^2} \int_{D(X_0, 3R)} |u - u_R|^2 dX$$

where

$$u_R = \frac{1}{|B(X_0, 2R)|} \int_{B(X_0, 2R)} u(X) dX.$$

In both cases, by Sobolev–Poincaré inequality, we obtain

$$\left(\frac{1}{|D(X_0, R)|} \int_{D(X_0, R)} |\nabla u|^2 dX \right)^{1/2} \leq C \left(\frac{1}{|D(X_0, 3R)|} \int_{D(X_0, 3R)} |\nabla u|^{6/5} dX \right)^{5/6}.$$

The lemma then follows from the usual reverse Hölder inequality (e.g., see [Gi, Proposition 1.1, p. 122]). $\qquad \blacksquare$

We now give the

Proof of Theorem 3.1.

Let $u \in \mathcal{D}(A)$ and $f = Au \in L^2_\sigma(\Omega)$. Fix $X_0 \in \Omega$. Let $D_t = B(X_0, t) \cap \Omega$ where $t \in (0, 1)$ is to be chosen later.

Let $f_1 = f$ in D_t and 0 otherwise. Let $f_2 = f - f_1$. We use (u_i, π_i), $i = 1, 2$ to denote solutions of (DP) for the Stokes equations with data f_i. By uniqueness, $u = u_1 + u_2$.

First we estimate $u_1(X_0)$. We claim that

$$|u_1(X_0)| \leq Ct^{1/2}\|f\|_{L^2(\Omega)}.$$

Indeed, for any $X \in D_t$, by the imbedding theorem of Morrey [GT, Theorem 7.17, p. 163],

$$|u_1(X) - u_1(X_0)| \leq Ct^\gamma \|\nabla u_1\|_{L^p(\Omega)}$$

where $p > 3$ and $\gamma = 1 - 3/p$. It then follows from Theorem 2.9 that

$$|u_1(X) - u_1(X_0)| \leq Ct^\gamma\|f_1\|_{W^{-1,p}(\Omega)} \leq Ct^\gamma\|f_1\|_{L^q(\Omega)} \leq t^{1/2}\|f\|_{L^2(\Omega)}$$

where $1/q = 1/p + 1/3$ and we also used the Sobolev imbedding theorem [GT, Theorem 7.26, p. 171] and Hölder inequality (we also assume that $p < 6$).

Now we integrate above inequality in X over D_t. This, together with the Sobolev imbedding, gives

$$
\begin{aligned}
|u_1(X_0)| &\leq \left(\frac{1}{|D_t|}\int_{D_t}|u_1|^6 dX\right)^{1/6} + Ct^{1/2}\|f\|_{L^2(\Omega)} \\
&\leq Ct^{-1/2}\|\nabla u_1\|_{L^2(\Omega)} + Ct^{1/2}\|f\|_{L^2(\Omega)} \\
&\leq Ct^{-1/2}\|f_1\|_{W^{-1,2}(\Omega)} + Ct^{1/2}\|f\|_{L^2(\Omega)} \\
&\leq Ct^{-1/2}\|f_1\|_{L^{6/5}(\Omega)} + Ct^{1/2}\|f\|_{L^2(\Omega)} \\
&\leq Ct^{1/2}\|f\|_{L^2(\Omega)}.
\end{aligned}
$$

To estimate $u_2(X_0)$, let $s \in [t/8, t/4]$ and $X \in D_s$. Note that $-\Delta u_2 + \nabla \pi_2 = 0$ and $\mathrm{div}\, u_2 = 0$ in D_s. We may apply Theorem 2.3 on D_s to obtain

$$
\begin{aligned}
|u_2(X) - u_2(X_0)| &\leq Ct^\alpha \sup_{\substack{P_1, P_2 \in \partial D_s \\ P_1 \neq P_2}} \frac{|u_2(P_1) - u_2(P_2)|}{|P_1 - P_2|^\alpha} \\
&\leq Ct^\alpha \left(\int_{\partial D_s}|\nabla u_2|^{p_0}d\sigma\right)^{1/p_0},
\end{aligned}
$$

where $p_0 = 2/(1-\alpha) > 2$ and we have used the Sobolev imbedding on the set ∂D_s.

By integration in s over $[t/8, t/4]$, we get

$$|u_2(X) - u_2(X_0)| \leq Ct^{\alpha - 1/p_0} \left(\int_{D_{t/4}} |\nabla u_2|^{p_0} dX \right)^{1/p_0}$$

$$\leq Ct \left(\frac{1}{|D_{t/4}|} \int_{D_{t/4}} |\nabla u_2|^{p_0} dX \right)^{1/p_0}$$

$$\leq Ct \left(\frac{1}{|D_t|} \int_{D_t} |\nabla u_2|^2 dX \right)^{1/2}$$

$$\leq Ct \left\{ \left(\frac{1}{|D_t|} \int_{D_t} |\nabla u|^2 dX \right)^{1/2} + \left(\frac{1}{|D_t|} \int_{D_t} |\nabla u_1|^2 dX \right)^{1/2} \right\}$$

$$\leq Ct^{-1/2} \|\nabla u\|_{L^2(\Omega)} + Ct^{-1/2} \|\nabla u_1\|_{L^2(\Omega)}$$

$$\leq Ct^{-1/2} \|\nabla u\|_{L^2(\Omega)} + Ct^{1/2} \|f\|_{L^2(\Omega)}$$

where we used the reverse Hölder inequality (Lemma 3.4) in the third inequality. Thus,

$$|u_2(X_0)| \leq \left(\frac{1}{|D_t|} \int_{D_t} |u_2|^6 dX \right)^{1/6} + Ct^{-1/2} \|\nabla u\|_{L^2(\Omega)} + Ct^{1/2} \|f\|_{L^2(\Omega)}$$

$$\leq Ct^{-1/2} \|u_2\|_{L^6(\Omega)} + Ct^{-1/2} \|\nabla u\|_{L^2(\Omega)} + Ct^{1/2} \|f\|_{L^2(\Omega)}$$

$$\leq Ct^{-1/2} \|\nabla u_2\|_{L^2(\Omega)} + Ct^{-1/2} \|\nabla u\|_{L^2(\Omega)} + Ct^{1/2} \|f\|_{L^2(\Omega)}$$

$$\leq Ct^{-1/2} \|\nabla u\|_{L^2(\Omega)} + Ct^{-1/2} \|\nabla u_1\|_{L^2(\Omega)} + Ct^{1/2} \|f\|_{L^2(\Omega)}$$

$$\leq Ct^{-1/2} \|\nabla u\|_{L^2(\Omega)} + Ct^{1/2} \|f\|_{L^2(\Omega)}.$$

To summarize, we have proved that

$$|u(X_0)| \leq Ct^{-1/2} \|\nabla u\|_{L^2(\Omega)} + Ct^{1/2} \|f\|_{L^2(\Omega)}$$

for any $t \in (0, c_0)$, where c_0 depends only on Ω.

Finally, by the energy estimate, $\|\nabla u\|_{L^2(\Omega)} \leq C_0 \|f\|_{L^2(\Omega)}$, so we may choose

$$t = \frac{c_0 \|\nabla u\|_{L^2(\Omega)}}{2C_0 \|f\|_{L^2(\Omega)}} < c_0$$

to obtain

$$|u(X_0)| \leq C \|\nabla u\|_{L^2(\Omega)}^{1/2} \|Au\|_{L^2(\Omega)}^{1/2}.$$

The proof is finished.

Appendix A

In this appendix, we present a simple proof of the area integral estimates for solutions of Stokes equations.

Theorem A.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. Suppose $\Delta u = \nabla \pi$, $div\, u = 0$ in Ω, and $(u)^* \in L^2(\partial\Omega)$. Then*

$$\int_\Omega |\nabla u(X)|^2 \delta(X) dX \leq C \int_{\partial\Omega} |u|^2 d\sigma$$

and

$$\int_{\partial\Omega} |u|^2 d\sigma \leq C \left\{ \int_\Omega |\nabla u(X)|^2 \delta(X) dX + \int_\Omega |u(X)|^2 dX \right\}$$

where $\delta(X) = dist(X, \partial\Omega)$.

Theorem A.1 is due to E. Fabes, C. Kenig and G. Verchota (unpublished). The proof given here is based on an idea of B. Dahlberg, C. Kenig, J. Pipher and G. Verchota [DKPV] developed for elliptic systems, and some observations on the pressure term π.

The following lemma is due to C. Kenig and E. Stein.

Lemma A.2. *Suppose $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function. Let $\theta \in C_0^\infty(\mathbb{R}^{n-1})$ be radial, and $\theta \geq 0$, $\int \theta dx = 1$. Then, if $\lambda \geq C(n, \|\nabla \psi\|_{L^\infty})$,*

$$(x, t) \to (x, y) = (x, \phi(x, t))$$

is a bi-Lipschitz map from $\mathbb{R}_+^n$ to $D = \{(x, y) : y > \psi(x)\}$ where $x \in \mathbb{R}^{n-1}$, t, $y \in \mathbb{R}$ and

$$\phi(x, t) = \lambda t + (\theta_t * \psi)(x), \quad \theta_t(x) = \frac{1}{t^{n-1}} \theta(\frac{x}{t}).$$

Moreover, $|\nabla^2 \phi(x, t)|^2 t\, dx\, dt$ is a Carleson measure on $\mathbb{R}_+^n$.

Lemma A.3. *Let $D = \{(x, y) \in \mathbb{R}^n : y > \psi(x)\}$ where $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function. Suppose that*

$$\begin{cases} -\Delta u + \nabla \pi = f \\ div\, u = g \end{cases}$$

in D. Also assume that u, π have compact supports and $(u)^* \in L^2(\partial D)$. Then

$$\int_D |\nabla u|^2 \delta(X) dX \leq C \left\{ \int_{\partial D} |(u)^*|^2 d\sigma + \int_D |\pi|^2 \delta(X) dX + \int_D |\nabla \pi|^2 \delta^3(X) dX \right.$$

$$\left. + \int_D |\Delta g + div\, f| |u| \delta^2(X) dX + \int_D |f| |u| \delta(X) dX \right\}$$

and

$$\int_{\partial D} |u|^2 d\sigma \leq \varepsilon \int_{\partial D} |(u)^*|^2 d\sigma + C_\varepsilon \left\{ \int_D |\nabla u|^2 \delta(X) dX + \int_D |\pi|^2 \delta(X) dX \right.$$

$$\left. + \int_D |\nabla \pi|^2 \delta^3(X) dX + \int_D |\Delta g + div\, f| |u| \delta^2(X) dX + \int_D |f| |u| \delta(X) dX \right\}$$

for any $\varepsilon > 0$.

Proof. Let $\rho : \mathbb{R}^n_+ \to D$ be defined by

$$\rho(x,t) = (x,y) = (x, \phi(x,t))$$

where $\phi(x,t) = \lambda t + \theta_t * \psi(x)$ is given in Lemma A.2.

Let $v = u \circ \rho$, $q = \pi \circ \rho$ be defined on $\mathbb{R}^n_+$. Note that v and q have compact supports in $\mathbb{R}^n_+$ and satisfy

(A.4)
$$
\begin{cases}
\dfrac{\partial v}{\partial t} = \dfrac{\partial u}{\partial y} \circ \rho \cdot \dfrac{\partial \phi}{\partial t} \\[2ex]
\dfrac{\partial v}{\partial x_i} = \dfrac{\partial u}{\partial x_i} \circ \rho + \dfrac{\partial u}{\partial y} \circ \rho \cdot \dfrac{\partial \phi}{\partial x_i}, \quad i = 1, 2, \ldots, n-1.
\end{cases}
$$

It follows from integration by parts that

$$\int_{R^{n-1}} |u \circ \rho|^2 \, dx = \int_{R^{n-1}} |v|^2 \, dx = -2 \iint_{\mathbb{R}^n_+} \frac{\partial v}{\partial t} \cdot v \, dx dt$$

(A.5)
$$= 2 \iint_{\mathbb{R}^n_+} t \left| \frac{\partial v}{\partial t} \right|^2 dx dt + 2 \iint_{\mathbb{R}^n_+} t \frac{\partial^2 v}{\partial t^2} \cdot v \, dx dt$$

$$= 2 \iint_{\mathbb{R}^n_+} t \left| \frac{\partial \phi}{\partial t} \right|^2 \left| \frac{\partial u}{\partial y} \circ \rho \right|^2 dx dt + 2 \iint_{\mathbb{R}^n_+} t \left| \frac{\partial \phi}{\partial t} \right|^2 \frac{\partial^2 u}{\partial y^2} \circ \rho \cdot u \circ \rho \, dx dt + G,$$

where G denotes terms which are bounded in absolute values by

$$C \iint_{\mathbb{R}^n_+} t |\nabla^2 \phi| \, |\nabla u \circ \rho| \, |u \circ \rho| \, dx dt + C \iint_{\mathbb{R}^n_+} t |\nabla^2 \phi| \, |\pi \circ \rho| \, |u \circ \rho| \, dx dt$$

(A.6)
$$+ C \iint_{\mathbb{R}^n_+} t^2 |\nabla^2 \phi| \, |\nabla \pi \circ \rho| \, |u \circ \rho| \, dx dt + C \iint_{\mathbb{R}^n_+} t |\pi \circ \rho| \, |\nabla u \circ \rho| \, dx dt$$

$$+ C \iint_{\mathbb{R}^n_+} t^2 |\nabla \pi \circ \rho| \, |\nabla u \circ \rho| \, dx dt.$$

By the Stokes equations $\Delta u = \nabla \pi - f$, we have

$$\frac{\partial^2 u}{\partial y^2} \circ \rho = -\frac{\partial^2 u}{\partial x_i^2} \circ \rho + \nabla \pi \circ \rho - f \circ \rho$$

$$= -\frac{\partial}{\partial x_i} \left\{ \frac{\partial u}{\partial x_i} \circ \rho \right\} + \frac{\partial^2 u}{\partial y \partial x_i} \circ \rho \cdot \frac{\partial \phi}{\partial x_i} + \nabla \pi \circ \rho - f \circ \rho$$

$$= -\frac{\partial}{\partial x_i} \left\{ \frac{\partial u}{\partial x_i} \circ \rho \right\} + \frac{\partial}{\partial x_i} \left\{ \frac{\partial u}{\partial y} \circ \rho \right\} \cdot \frac{\partial \phi}{\partial x_i} - \frac{\partial^2 u}{\partial y^2} \circ \rho \cdot |\nabla_x \phi|^2 + \nabla \pi \circ \rho - f \circ \rho$$

where the repeated index i is summed over $\{1, 2, \ldots, n-1\}$ and $\nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_{n-1})$.
It follows that

$$\frac{\partial^2 u}{\partial y^2} \circ \rho$$

$$= \frac{1}{1 + |\nabla_x \phi|^2} \left\{ -\frac{\partial}{\partial x_i} \left\{ \frac{\partial u}{\partial x_i} \circ \rho \right\} + \frac{\partial}{\partial x_i} \left\{ \frac{\partial u}{\partial y} \circ \rho \right\} \cdot \frac{\partial \phi}{\partial x_i} + \nabla \pi \circ \rho - f \circ \rho \right\}.$$

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Thus, we may use integration by parts to obtain

$$\iint_{\mathbb{R}^n_+} t|\frac{\partial \phi}{\partial t}|^2 \frac{\partial^2 u}{\partial y^2} \circ \rho \cdot u \circ \rho \, dx dt$$

$$= \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} \cdot \frac{\partial u}{\partial x_i} \circ \rho \cdot \frac{\partial}{\partial x_i} \{u \circ \rho\} \, dx dt$$

$$- \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2 \frac{\partial \phi}{\partial x_i}}{1 + |\nabla_x \phi|^2} \cdot \frac{\partial u}{\partial y} \circ \rho \cdot \frac{\partial}{\partial x_i} \{u \circ \rho\} \, dx dt$$

$$+ \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} \cdot \{\nabla \pi \circ \rho - f \circ \rho\} \cdot u \circ \rho \, dx dt + G$$

$$= \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} \cdot |\frac{\partial u}{\partial x_i} \circ \rho|^2 \, dx dt$$

$$- \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2 |\nabla_x \phi|^2}{1 + |\nabla_x \phi|^2} \cdot |\frac{\partial u}{\partial y} \circ \rho|^2 \, dx dt$$

$$+ \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} \cdot \{\nabla \pi \circ \rho - f \circ \rho\} \cdot u \circ \rho \, dx dt + G.$$

This, together with (A.5), implies that

(A.7)
$$\int_{\mathbb{R}^{n-1}} |u \circ \rho|^2 dx = 2 \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} \cdot |\nabla u \circ \rho|^2 \, dx dt$$

$$+ 2 \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} \cdot \{\nabla \pi \circ \rho - f \circ \rho\} \cdot u \circ \rho \, dx dt + G.$$

It remains to estimate

$$\iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} \nabla \pi \circ \rho \cdot u \circ \rho \, dx dt.$$

To this end, we let $u = (u_1, u_2, \ldots, u_n)$ and note that

$$\nabla \pi \circ \rho \cdot u \circ \rho = \frac{\partial \pi}{\partial x_i} \circ \rho \cdot u_i \circ \rho + \frac{\partial \pi}{\partial y} \circ \rho \cdot u_n \circ \rho$$

$$= \frac{\partial}{\partial x_i} \{\pi \circ \rho\} \cdot u_i \circ \rho + \frac{\partial \pi}{\partial y} \circ \rho \cdot \left\{ u_n \circ \rho - u_i \circ \rho \frac{\partial \phi}{\partial x_i} \right\}.$$

Thus, the integration by parts yields

(A.8)
$$2 \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} \nabla \pi \circ \rho \cdot u \circ \rho \, dx dt$$

$$= 2 \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} \frac{\partial \pi}{\partial y} \circ \rho \cdot \left\{ u_n \circ \rho - u_i \circ \rho \frac{\partial \phi}{\partial x_i} \right\} \, dx dt + G.$$

Now, we write $2t = \frac{\partial t^2}{\partial t}$. Using integration by parts, we obtain

$$2 \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} \cdot \frac{\partial \pi}{\partial y} \circ \rho \cdot \left\{ u_n \circ \rho - u_i \circ \rho \frac{\partial \phi}{\partial x_i} \right\} dx dt$$

$$= -\iint_{\mathbb{R}^n_+} \frac{t^2(\frac{\partial \phi}{\partial t})^3}{1 + |\nabla_x \phi|^2} \cdot \frac{\partial^2 \pi}{\partial y^2} \circ \rho \cdot \left\{ u_n \circ \rho - u_i \circ \rho \frac{\partial \phi}{\partial x_i} \right\} dx dt + G.$$

Since $\Delta u = \nabla \pi - f$, we have $\Delta \pi = \Delta(\operatorname{div} u) + \operatorname{div} f = \Delta g + \operatorname{div} f$. Thus,

$$\frac{\partial^2 \pi}{\partial y^2} \circ \rho = -\frac{\partial^2 \pi}{\partial x_i^2} \circ \rho + \Delta g \circ \rho + (\operatorname{div} f) \circ \rho$$

$$= -\frac{\partial}{\partial x_i} \left\{ \frac{\partial \pi}{\partial x_i} \circ \rho \right\} + \frac{\partial}{\partial x_i} \left\{ \frac{\partial \pi}{\partial y} \circ \rho \right\} \cdot \frac{\partial \phi}{\partial x_i} - \frac{\partial^2 \pi}{\partial y^2} \circ \rho \, |\nabla_x \phi|^2$$

$$+ \Delta g \circ \rho + (\operatorname{div} f) \circ \rho.$$

It follows that

$$\frac{\partial^2 \pi}{\partial y^2} \circ \rho =$$

$$\frac{1}{1 + |\nabla_x \phi|^2} \left\{ -\frac{\partial}{\partial x_i} \left\{ \frac{\partial \pi}{\partial x_i} \circ \rho \right\} + \frac{\partial}{\partial x_i} \left\{ \frac{\partial \pi}{\partial y} \circ \rho \right\} \cdot \frac{\partial \phi}{\partial x_i} + \Delta g \circ \rho + (\operatorname{div} f) \circ \rho \right\}.$$

Hence, the integration by parts again yields

$$2 \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} \cdot \frac{\partial \pi}{\partial y} \circ \rho \cdot \left\{ u_n \circ \rho - u_i \circ \rho \frac{\partial \phi}{\partial x_i} \right\} dx dt$$

$$= -\iint_{\mathbb{R}^n_+} \frac{t^2(\frac{\partial \phi}{\partial t})^3}{(1 + |\nabla_x \phi|^2)^2} \{\Delta g + \operatorname{div} f\} \circ \rho \cdot \left\{ u_n \circ \rho - u_i \circ \rho \frac{\partial \phi}{\partial x_i} \right\} dx dt + G.$$

Thus, putting together (A.7), (A.8) and the above estimate, we have proved that

$$\int_{\mathbb{R}^{n-1}} |u \circ \rho|^2 dx = 2 \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} |\nabla u \circ \rho|^2 \, dx dt$$

$$- \iint_{\mathbb{R}^n_+} \frac{t^2(\frac{\partial \phi}{\partial t})^3}{(1 + |\nabla_x \phi|^2)^2} \{\Delta g + \operatorname{div} f\} \circ \rho \cdot \left\{ u_n \circ \rho - u_i \circ \rho \frac{\partial \phi}{\partial x_i} \right\} dx dt$$

$$- 2 \iint_{\mathbb{R}^n_+} \frac{t(\frac{\partial \phi}{\partial t})^2}{1 + |\nabla_x \phi|^2} f \circ \rho \cdot u \circ \rho \, dx dt + G.$$

This implies that

$$\int_{\partial D} |u|^2 d\sigma \leq C \int_D |\nabla u|^2 \delta(X) dX + C \int_D |\Delta g + \operatorname{div} f| \, |u| \delta^2(X) dX$$

$$+ C \int_D |f| \, |u| \delta(X) dX + |G|$$

and

$$\int_D |\nabla u|^2 \delta(X) dX \leq C \int_{\partial D} |u|^2 d\sigma + C \int_D |\Delta g + \operatorname{div} f| |u| \delta^2(X) dX$$
$$+ C \int_D |f| |u| \delta(X) dX + |G|.$$

Since $|\nabla^2 \phi|^2 t dx dt$ is a Carleson measure, by Cauchy-Schwartz inequality and (A.6), we have

$$|G| \leq C \left(\int_D |\nabla u|^2 \delta(X) dX \right)^{1/2} \left(\int_{\partial D} |(u)^*|^2 d\sigma \right)^{1/2}$$
$$+ C \left(\int_D |\pi|^2 \delta(X) dX \right)^{1/2} \left(\int_{\partial D} |(u)^*|^2 d\sigma \right)^{1/2}$$
$$+ C \left(\int_D |\nabla \pi|^2 \delta^3(X) dX \right)^{1/2} \left(\int_{\partial D} |(u)^*|^2 d\sigma \right)^{1/2}$$
$$+ C \left(\int_D |\pi|^2 \delta(X) dX \right)^{1/2} \left(\int_D |\nabla u|^2 \delta(X) dX \right)^{1/2}$$
$$+ C \left(\int_D |\nabla \pi|^2 \delta^3(X) dX \right)^{1/2} \left(\int_D |\nabla u|^2 \delta(X) dX \right)^{1/2}.$$

The lemma now follows easily from the Cauchy inequality.

Lemma A.9. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. Suppose that $\Delta u = \nabla \pi$, $\operatorname{div} u = 0$ in Ω, and $(u)^* \in L^2(\partial \Omega)$. Then there exists a function $\tilde{\pi}$ such that $\tilde{\pi} = \pi + c$ and*

$$\int_\Omega |\tilde{\pi}(X)|^2 \delta(X) dX + \int_\Omega |\nabla \tilde{\pi}(X)|^2 \delta^3(X) dX \leq C \int_{\partial \Omega} |u|^2 d\sigma.$$

Proof. By the L^2 estimates in [FKV], u can be represented in terms of a double layer potential:

$$u_i(X) = \int_{\partial \Omega} \left\{ \frac{\partial}{\partial y_k} \{\Gamma_{ij}(X - Y)\} N_k(Y) - q^i(X - Y) N_j(Y) \right\} h_j(Y) d\sigma(Y)$$

where $\|h\|_{L^2(\partial \Omega)} \leq C \|u\|_{L^2(\partial \Omega)}$, $(\Gamma_{ij}(X))$ is the matrix of fundamental solutions and $(q^i(X))$ is the corresponding pressure vector given in (2.8).

Note that

$$\Delta \Gamma_{ij}(X) = \frac{1}{2\omega_n} \Delta \left\{ \frac{x_i x_j}{|X|^n} \right\} = \frac{1}{\omega_n} \nabla \left(\frac{x_i}{|X|^n} \right) \cdot \nabla x_j = \frac{1}{\omega_n} \frac{\partial}{\partial x_i} \left\{ \frac{x_j}{|X|^n} \right\}.$$

Thus,

$$\Delta u_i(X) = \Delta \int_{\partial \Omega} \frac{\partial}{\partial y_k} \{\Gamma_{ij}(X - Y)\} N_k(Y) h_j(Y) d\sigma(Y)$$
$$= -\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} \int_{\partial \Omega} \frac{x_j - y_j}{\omega_n |X - Y|^n} N_k(Y) h_j(Y) d\sigma(Y)$$
$$= \frac{\partial \tilde{\pi}}{\partial x_i}$$

where

$$\tilde{\pi}(X) = -\frac{\partial w_k(X)}{\partial x_k}, \quad \text{and} \quad w_k(X) = \int_{po} \frac{x_j - y_j}{\omega_n |X - Y|^n} N_k(Y) h_j(Y) \, d\sigma(Y).$$

Clearly, $\tilde{\pi} = \pi + c$ since $\nabla \tilde{\pi} = \nabla \pi$. Note that w_k is harmonic in Ω and $(w_k)^* \in L^2(\partial \Omega)$. By the area integral estimates for the harmonic function [D],

$$\int_\Omega |\tilde{\pi}|^2 \delta(X) dX \leq \sum_{k=1}^n \int_\Omega |\nabla w_k|^2 \delta(X) dX \leq C \sum_{k=1}^n \int_{\partial \Omega} |(w_k)^*|^2 d\sigma$$

$$\leq C \int_{\partial \Omega} |h|^2 d\sigma \leq C \int_{\partial \Omega} |u|^2 d\sigma.$$

Also, since $\Delta \tilde{\pi} = 0$ in Ω, by interior estimates,

$$\int_\Omega |\nabla \tilde{\pi}(X)|^2 \delta^3(X) dX \leq C \int_\Omega |\tilde{\pi}(X)|^2 \delta(X) dX \leq C \int_{\partial \Omega} |u|^2 d\sigma.$$

This completes the proof.

We are now in a position to give the
Proof of Theorem A.1.
Fix $P \in \partial \Omega$ and $r > 0$ small. Using linear transformations both in the variable X and functions u, π (see [S1, p. 347]), we may assume that

$$\Omega \cap B(P, 3r) = \{(x, y) \in \mathbb{R}^n : y > \psi(x)\} \cap B(P, 3r).$$

Let $\eta \in C_0^\infty(B(P, 2r))$ such that $\eta \equiv 1$ on $B(P, r)$. Since $\Delta u = \nabla \pi$, $\operatorname{div} \pi = 0$ in Ω, we have

$$\Delta(u\eta) = (\Delta u)\eta + 2\nabla u \cdot \nabla \eta + u \cdot \Delta \eta$$
$$= (\nabla \pi)\eta + 2\nabla u \cdot \nabla \eta + u \cdot \Delta \eta$$
$$= \nabla(\pi \eta) - \pi \nabla \eta + 2\nabla u \cdot \nabla \eta + u \cdot \Delta \eta$$
$$= \nabla(\pi \eta) - f$$

where $f = \pi \nabla \eta - 2\nabla u \cdot \nabla \eta - u \cdot \Delta \eta$.

Note that $|f| \leq C\{|\pi| + |\nabla u| + |u|\}$. Also, $\operatorname{div}(u\eta) = u \cdot \nabla \eta \equiv g$ and $|g| \leq C|u|$. Moreover, it is not hard to see that

$$|\Delta g + \operatorname{div} f| \leq C\{|\nabla u| + |\nabla \pi| + |\pi| + |u|\}.$$

We now apply Lemma A.3 to the equations

$$-\Delta(u\eta) + \nabla(\pi \eta) = f, \quad \operatorname{div}(u\eta) = g$$

in $D = \{(x, y) \in \mathbb{R}^n : y > \psi(x)\}$. Let $D_r = \Omega \cap B(P, r)$. We obtain

$$\int_{D_r} |\nabla u|^2 \delta(X) dX \leq C \int_{\partial \Omega} |u|^2 d\sigma + C \int_\Omega |\pi|^2 \delta(X) dX + C \int_\Omega |\nabla \pi|^2 \delta^3(X) dX$$

$$+ C \int_\Omega |\nabla u| |u| \delta^2(X) dX + C \int_\Omega |u|^2 dX$$

$$\leq C \int_{\partial \Omega} |u|^2 d\sigma,$$

where we have used Lemma A.9 in the second inequality. It follows that

$$\int_\Omega |\nabla u|^2 \delta(X) dX \le C \int_{\partial\Omega} |u|^2 d\sigma.$$

To finish the proof, note that Lemma A.3 also gives

$$\int_{\partial\Omega \cap B(P,r)} |u|^2 d\sigma \le \varepsilon \int_{\partial\Omega} |(u)^*|^2 d\sigma + C_\varepsilon \left\{ \int_\Omega |\nabla u|^2 \delta(X) dX + \int_\Omega |u|^2 dX \right.$$
$$\left. \int_\Omega |\pi|^2 \delta(X) dX + \int_\Omega |\nabla \pi|^2 \delta^3(X) dX \right\}.$$

It then follows from the L^2-estimates [FKV] that

$$\int_{\partial\Omega} |u|^2 d\sigma \le C \left\{ \int_\Omega |\nabla u|^2 \delta(X) dX + \int_\Omega |\pi|^2 \delta(X) dX + \int_\Omega |u|^2 dX \right\}.$$

This estimate holds for any pressure term π. In particular, we may choose π such that $\pi(X_0) = 0$ for some $X_0 \in \Omega$. Then, by the Hardy inequality [St, p. 272],

$$\int_\Omega |\pi|^2 \delta(X) dX \le C \int_\Omega |\nabla \pi|^2 \delta^3(X) dX \le C \int_\Omega |\nabla^2 u|^2 \delta^3(X) dX$$
$$\le C \int_\Omega |\nabla u|^2 \delta(X) dX$$

where the last inequality follows from the interior estimates.

Thus,

$$\int_{\partial\Omega} |u|^2 d\sigma \le C \left\{ \int_\Omega |\nabla u|^2 \delta(X) dX + \int_\Omega |u|^2 dX \right\}.$$

The proof is complete.

References

[BL]   J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Springer-Verlag, Berlin, 1976.

[CF]   P. Constantin and C. Foias, *Navier-Stokes Equations*, Univ. of Chicago Press, 1988.

[D]   B. Dahlberg, *Weighted norm inequalities for the Lusin area integral and the nontangential maximal functions for harmonic functions in a Lipschitz domain*, Studia Math. **67** (1980), 297-314.

[DKPV]   ———, C. Kenig, J. Pipher ,and G. Verchota, *Area integral estimates and maximum principles for higher order elliptic equations and systems on Lipschitz domains*, preprint.

[DW]   P. Deuring and W. von Wahl, *Strong solutions of the Navier-Stokes system on Lipschitz bounded domains*, Math. Nachr. **171** (1995), 111-148.

[F]   E. Fabes, *Layer potential methods for boundary value problems on Lipschitz domains*, Lecture Notes in Math. **1344** (1988), 55-80.

[FKV]   ———, C. Kenig and G. Verchota, *The Dirichlet problem for the Stokes system on Lipschitz domains*, Duke Math. J. **57** (1988), 769-793.

[FK]   H. Fujita and T. Kato, *On the Navier-Stokes initial value problem*, Arch. Rat. Mech. Anal. **16** (1964), 269-315.

[Gi]   M. Giaquinta, *Multiple Integrals in the Calculus of Variation and Nonlinear Elliptic Systems*, Princeton Univ. Press, 1983.

[GT]   D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, Springer-Verlag, 1983.

[Gr]   P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, 1985.

[H]   J. G. Heywood, *The Navier-Stokes equations: on the existence, regularity and decay of solutions*, Indiana Univ. Math. J. **29** (1980), 639-681.

[JK]   D. Jerison and C. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Functional Analysis **130** (1995), 161-219.

[S1]   Z. Shen, *Boundary value problems for parabolic Lamé systems and a nonstationary linearized system of Navier-Stokes equations in Lipschitz cylinders*, Amer. J. Math. **113** (1991), 293-373.

[S2]   _____, *A note on the Dirichlet problem for the Stokes system in Lipschitz domains*, Proc. Amer. Math. Soc. **123(3)** (1995), 801-811.

[St]   E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1971.

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