Quasisymmetric refinements of Schur functions

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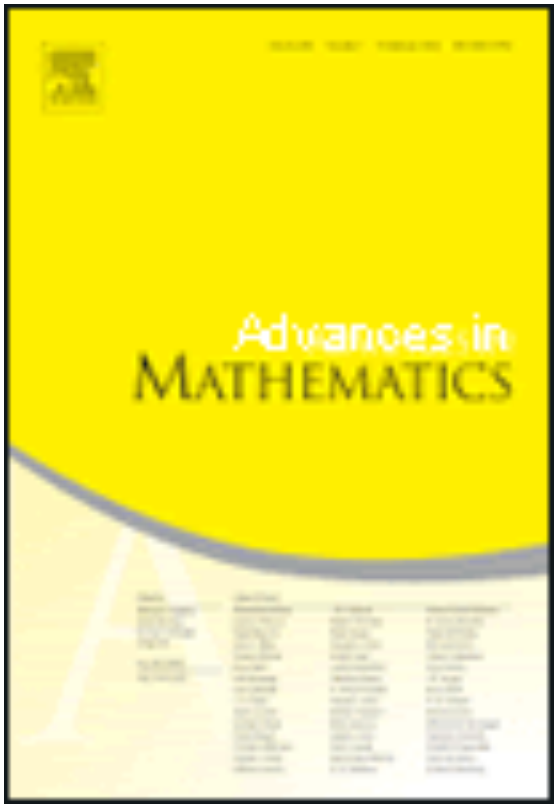
Combinatorics: Advances



Compositions and partitions

A composition $\alpha_1 \ldots \alpha_k$ of n is a list of positive integers whose sum is n: $2213 \models 8$.

A composition is a partition if $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k > 0$: $3221 \vdash 8$.

Any composition determines a partition: $\lambda(2213) = 3221$.

$\alpha = \alpha_1 \ldots \alpha_k$ is a coarsening of $\beta = \beta_1 \ldots \beta_l$ (β is a refinement of α) if

$$\underbrace{\beta_1 + \ldots + \beta_i}_{\alpha_1} \underbrace{\beta_{i+1} + \ldots + \beta_j}_{\alpha_2} \ldots \underbrace{\beta_m + \ldots + \beta_l}_{\alpha_k}$$

is true: $53 \succcurlyeq 2213$.

Symmetric functions

Let Sym be the algebra of symmetric functions

$$Sym := Sym_0 \oplus Sym_1 \oplus \cdots \subset \mathbb{Q}[x_1, x_2, \ldots]$$

$$Sym_n := \text{span}_{\mathbb{Q}}\{m_\lambda \mid \lambda = \lambda_1 \ldots \lambda_k \vdash n\} = \text{span}_{\mathbb{Q}}\{h_\lambda \mid \lambda \vdash n\}$$

$$m_\lambda := \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k} \text{ distinct}$$

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k} \qquad h_r := \sum_{i_1 \leq i_2 \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Example $m_{211} = x_1^2 x_2 x_3 + x_2^2 x_1 x_3 + x_3^2 x_1 x_2 \ldots$

Why symmetric functions?

- Generating function for tableaux.
- Representation theory of Lie algebras.
- Representation theory of $S_n, GL(v, \mathbb{C})$.
- Schubert calculus as certain Schubert polynomials.
- Noncommutative and nonsymmetric analogues.

Noncommutative symmetric functions

Let $NSym$ be the noncommutative symmetric functions

$$NSym := NSym_0 \oplus NSym_1 \oplus \cdots \subset \mathbb{Q}\langle x_1, x_2, \ldots \rangle$$

$$NSym_n := \mathrm{span}_{\mathbb{Q}}\{\mathbf{h}_\alpha \mid \alpha = \alpha_1 \ldots \alpha_k \vDash n\} = \mathrm{span}_{\mathbb{Q}}\{R_\alpha \mid \alpha \vDash n\}$$

$$\mathbf{h}_\alpha := \mathbf{h}_{\alpha_1}\mathbf{h}_{\alpha_2}\cdots\mathbf{h}_{\alpha_k} \qquad \mathbf{h}_r := \sum_{\sum \beta_i = r} (-1)^{\ell(\beta)-r} x_{\beta_1} x_{\beta_2} \cdots x_{\beta_{\ell(\beta)}}$$

$$R_\alpha := \sum_{\beta \succcurlyeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \mathbf{h}_\beta$$

Remark $NSym \to Sym$ via variables commute $\mathbf{h}_\alpha \mapsto h_{\lambda(\alpha)}$.

Why noncommutative symmetric functions?

• Anti-isomorphic to Solomon's descent algebra (GKLLRT 95, Malvenuto-Reutenauer 95).

• Study of riffle shuffles (Bayer-Diaconis 92).

• Lie algebra idempotents (Garsia-Reutenauer 89).

• Transportation matrices (Garsia-Remmel 85).

• Representation theory of 0-Hecke algebra and elsewhere (overview Thibon 01).

Quasisymmetric functions

Let $QSym$ be the algebra of quasisymmetric functions

$$QSym := QSym_0 \oplus QSym_1 \oplus \cdots \subset \mathbb{Q}[x_1, x_2, \ldots]$$

$$QSym_n := \mathrm{span}_{\mathbb{Q}}\{M_\alpha \mid \alpha = \alpha_1 \ldots \alpha_k \vDash n\} = \mathrm{span}_{\mathbb{Q}}\{F_\alpha \mid \alpha \vDash n\}$$

$$M_\alpha := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \qquad F_\alpha = \sum_{\alpha \succcurlyeq \beta} M_\beta$$

Example $M_{121} = \sum_{i_1<i_2<i_3} x_{i_1}^1 x_{i_2}^2 x_{i_3}^1$, $F_{121} = M_{121} + M_{1111}$

Remark $Sym \hookrightarrow QSym$ via $m_\lambda = \sum_{\lambda(\alpha)=\lambda} M_\alpha$.

Why quasisymmetric functions?

• Generating functions for P-partitions, posets, matroids (Gessel 83, Ehrenborg 96, Luoto 09, Billera-Jia-Reiner 2009).

• Combinatorial Hopf algebras (Ehrenborg 96, Aguiar-Bergeron-Sottile 06).

• Dual to cd-index (Billera-Hsiao-vW 03).

• Random walks (Stanley 01, Hsiao-Hersh 09, Ehrenborg-Readdy).

• Simplify Macd., K-L polys (Haglund-Luoto-Mason-vW 09, Billera-Brenti).

• Other types, coloured, shifted (Billey-Haiman 95, Ehrenborg-Readdy).

Trends


VOGUE
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Beauty's new masterstrokes
CHARACTER THEORY
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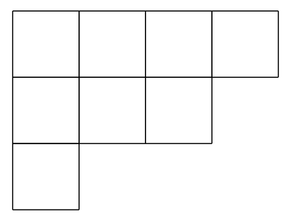
VICTORIA BECKHAM
Who's in your spiritual address book?

Diagrams and tableaux

The diagram $\lambda = \lambda_1 \geq \ldots \geq \lambda_k > 0$ is the array of boxes with λ_i boxes in row i from the top.



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A (standard) reverse tableau T of shape λ is a filling of λ with (each first n) $1, 2, 3, \ldots$ so rows weakly decrease and columns strictly decrease.

Diagrams and tableaux

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8	7	3	1
6	4	2	
5			

431

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Schur functions

If $x^T := x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \ldots$ then $Sym_n = \text{span}_{\mathbb{Q}}\{s_\lambda \mid \lambda \vdash n\}$ where

$$s_\lambda = \sum_{T \in RT(\lambda)} x^T$$

Example $s_{21} = x_1^2 x_2 + x_1 x_2^2 + 2x_1x_2x_3 + \cdots$ from

2	1
1	

2	2
1	

3	2
1	

3	1
2	

Schur function generalizations and analogues

- Hall-Littlewood and Macdonald polynomials.

- Schur P,Q functions, factorial Schur functions, cylindric Schur funtions, free Schur functions, k-Schur functions...

- Schur functions in noncommuting variables of Rosas/Sagan and Fomin/Greene.

- (Skew) Quasisymmetric Schur functions.

- Noncommutative Schur functions.

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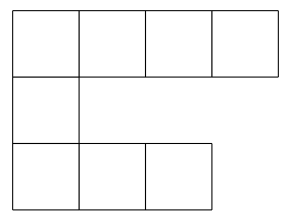
Composition diagrams and tableaux

The composition diagram $\alpha = \alpha_1 \ldots \alpha_k > 0$ is the array of boxes with α_i boxes in row i from the top.



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A (standard) composition tableau of shape α is a filling of α with (each first n) $1, 2, 3, \ldots$ such that

Rules for composition tableaux

- First column entries strictly increase top to bottom.
- Rows weakly decrease left to right.
- If $b \leq c$ then $b < a$.

Example

c	a
	b

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Rules for composition tableaux

- First column entries strictly increase top to bottom.

- Rows weakly decrease left to right.

- If $b \leq c$ then $b < a$.

Example

5	4	3	1
6			
8	7	2	

Quasisymmetric Schur functions

If $x^T := x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \ldots$ then $QSym_n = \mathrm{span}_{\mathbb{Q}}\{\mathcal{S}_\alpha \mid \alpha \vDash n\}$ where

$$\mathcal{S}_\alpha = \sum_{T \in CT(\alpha)} x^T$$

Example

Remark $s_\lambda = \sum_{\lambda(\alpha)=\lambda} \mathcal{S}_\alpha$ as $m_\lambda = \sum_{\lambda(\alpha)=\lambda} M_\alpha$.

Quasisymmetric Schur functions

If $x^T := x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \ldots$ then $QSym_n = \text{span}_{\mathbb{Q}}\{\mathcal{S}_\alpha \mid \alpha \vDash n\}$ where

$$\mathcal{S}_\alpha = \sum_{T \in CT(\alpha)} x^T$$

Example $\mathcal{S}_{12} = x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2$ from

1	
2	2

1	
3	2

1	
3	3

2	
3	3

Remark $s_\lambda = \sum_{\lambda(\alpha)=\lambda} \mathcal{S}_\alpha$ as $m_\lambda = \sum_{\lambda(\alpha)=\lambda} M_\alpha$.

Quasisymmetric Kostka numbers

For $\lambda \vdash n$

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

where $K_{\lambda\mu}$ = number of reverse tableaux T of shape λ and μ_1 1s, μ_2 2s, ...

For $\alpha \vDash n$

$$\mathcal{S}_\alpha = \sum_\beta K_{\alpha\beta} M_\beta$$

where $K_{\alpha\beta}$ = number of composition tableaux T of shape α and β_1 1s, β_2 2s, ...

Quasisymmetric L-R rule: via products

In Sym

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \qquad c^\lambda_{\mu\nu} : \text{+ve integers}$$

In Qsym

$$\mathcal{S}_\alpha \mathcal{S}_\beta = \sum_\lambda C^\gamma_{\alpha\beta} \mathcal{S}_\gamma \qquad C^\gamma_{\alpha\beta} : \text{some -ve integers}$$

Remark But $s_\mu \mathcal{S}_\alpha$ yields +ve integers in $\mathcal{S}_\gamma$ (Haglund-Mason-Luoto-vW 09).

Young's lattice: $\mathcal{L}_Y$

Partial order on partitions with covers

- add 1 at end: $211 < 2111$
- add 1 to leftmost part of size: $211 < 221$, $211 < 311$.

saturated chains in $\mathcal{L}_Y$ from μ to λ	$\leftrightarrow$	standard skew RT shape λ/μ

Example

$32 < 321 < 331 < 431 \leftrightarrow$

•	•	•	1
•	•	2	
3			

Composition poset: $\mathcal{L}_C$

Partial order on compositions with covers

- add 1 at start: $121 < 1121$
- add 1 to leftmost part of size: $121 < 221$, $121 < 131$.

saturated chains in $\mathcal{L}_C$ from α to β $\leftrightarrow$ standard skew CT shape $\beta//\alpha$

Example

$$23 < 123 < 133 < 143 \leftrightarrow$$

3			
•	•	2	1
•	•	•	

Descents and sets

T standard (skew) tableau, $Des(T) = \{i \mid i+1$ weakly east $\}$:

8	7	3	1
6	4	2	
5			

composition $\alpha_1 \ldots \alpha_k \vDash n \leftrightarrow$ subset $\{i_1, \ldots, i_{k-1}\} \subseteq [n-1]$

$\beta \qquad 2312 \vDash 8 \leftrightarrow \{2,5,6\} \subseteq [7] \qquad Set(\beta)$

Quasisymmetric L-R rule: via skew functions

Skew Schur functions

$$s_{\lambda/\mu} = \sum F_\delta = \sum_\nu c^\lambda_{\mu\nu} s_\nu \qquad c^\lambda_{\mu\nu} : \text{+ve integers}$$

where $Set(\delta) = Des(T)$, $T \in SRT(\lambda/\mu)$.

Skew quasisymmetric Schur functions

$$\mathcal{S}_{\gamma//\beta} = \sum F_\delta = \sum_\alpha C^\gamma_{\alpha\beta} \mathcal{S}_\alpha \qquad C^\gamma_{\alpha\beta} : \text{+ve integers}$$

where $Set(\delta) = Des(T)$, $T \in SCT(\gamma//\beta)$.

See the rule $\rightsquigarrow$ Kurt Luoto's talk!

Dual Hopf algebras: H and H^*

- Product in $H \leftrightarrow$ Coproduct in H^*.

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- Coproduct Δ defines skew elements $B_{i/j}$.

$$\{B_i\} \text{ basis of } H \Rightarrow \Delta B_i = \sum_j B_{i/j} \otimes B_j$$

Dual Hopf algebras: Sym and Sym

m_λ is dual to h_λ, s_λ is dual to itself. So

$$\Delta s_\lambda = \sum s_{\lambda/\mu} \otimes s_\mu = \sum c^\lambda_{\mu\nu} s_\nu \otimes s_\mu \Leftrightarrow s_\nu s_\mu = \sum c^\lambda_{\mu\nu} s_\lambda.$$

Dual Hopf algebras: QSym and NSym

M_α is dual to $\mathbf{h}_\alpha$, F_α is dual to R_α, $\mathcal{S}_\alpha$ is dual to $\mathcal{S}^*_\alpha$. So

$$\Delta \mathcal{S}_\gamma = \sum \mathcal{S}_{\gamma//\beta} \otimes \mathcal{S}_\beta = \sum C^\gamma_{\alpha\beta} \mathcal{S}_\alpha \otimes \mathcal{S}_\beta \Leftrightarrow \mathcal{S}^*_\alpha \mathcal{S}^*_\beta = \sum C^\gamma_{\alpha\beta} \mathcal{S}^*_\gamma.$$

Noncommutative Schur functions $\mathcal{S}^*_\alpha$ satisfy a noncommutative L-R rule.

Link to NC Schurs of Fomin and Greene

P graded edge labelled poset, labels $(B,<)$. For $x \in P$

$$x.\mathbf{h}_k = \sum_{\omega} end(\omega)$$

$$\omega : x \xrightarrow{b_1} x_1 \xrightarrow{b_2} \cdots \xrightarrow{b_k} x_k = end(\omega)$$

for saturated ω, $b_1 \leq b_2 \leq \cdots \leq b_k \in B$.

For $[x,y]$ of P

$$K_{[x,y]} = \sum_{\alpha} \langle x.\mathbf{h}_\alpha, y \rangle M_\alpha \qquad \langle \ , \ \rangle = \delta_{ij}$$

Example Skew Schur functions, Stanley symmetric functions, NC Schurs Fomin+Greene (Bergeron-Mykytiuk-Sottile-vW 00).

A new example

Let $\mathcal{L}'_C$ be the dual poset of $\mathcal{L}_C$ edges labelled

$$x \stackrel{(-col,-row)}{\longrightarrow} \tilde{x}$$

and $(i,j) < (k,\ell)$ iff $i < k$ or ($i = k$ and $j < \ell$).

Then

$$K_{[\beta,\alpha]} = \mathcal{S}_{\beta//\alpha}.$$

Link to NC Schurs of Rosas and Sagan

A set composition of $[n] = \{1, \ldots, n\}$ is an ordered partitioning of $[n]$: $\Phi = 36/489/2/157 \models [9]$ with underlying composition $\alpha(\Phi) = 2313$.

A set partition of $[n]$ reorders by least element: $\tilde{\Phi} = 157/2/36/489 \vdash [9]$ with underlying partition $\lambda(\Phi) = 3321$.

Symmetric functions in noncommuting variables

(Wolfe 36; Rosas-Sagan 06)

$$NCSym := NCSym_0 \oplus NCSym_1 \oplus \cdots \subset \mathbb{Q}\langle x_1, x_2, \ldots\rangle$$

where

$$NCSym_n := \text{span}_{\mathbb{Q}}\{\mathbf{m}_\pi \mid \pi \vdash [n]\}$$

$$\mathbf{m}_\pi := \sum x_{i_1} x_{i_2} \cdots x_{i_n} \text{ and } i_j = i_k \text{ iff } j, k \in \pi_m$$

Example $\mathbf{m}_{13/2} = x_1x_2x_1 + x_2x_1x_2 + x_1x_3x_1 + x_3x_1x_3 \ldots$

NC Schurs of Rosas and Sagan

For $T \in RT(\lambda)$ let $\dot{T}$ have 1 entry with k dots $k = 1, 2, 3 \ldots$ then

$$S_\lambda^{RS} = \sum_{T \in RT(\lambda)} x^{\dot{T}} = \sum_\mu \mu! K_{\lambda\mu} \sum_{\lambda(\pi)=\mu} \mathbf{m}_\pi$$

where $x^{\dot{T}}$ = monomial x_i in position j if T has i with j dots.

Example

$$\begin{array}{|c|c|}\hline \dot{2} & \dddot{1} \\ \hline \ddot{3} \\ \cline{1-1}\end{array} \qquad \rightsquigarrow \qquad x_2 x_3 x_1$$

Quasisymmetric functions in noncommuting variables

(Bergeron-Zabrocki)

$$NCSym \subset NCQSym := NCQSym_0 \oplus NCQSym_1 \oplus \cdots \subset \mathbb{Q}\langle x_1, x_2, \ldots \rangle$$

where

$$NCQSym_n := \text{span}_{\mathbb{Q}}\{\mathbf{M}_\Pi \mid \Pi \vDash [n]\}$$

$$\mathbf{M}_\Pi := \sum x_{i_1} x_{i_2} \cdots x_{i_n}$$

- $i_j = i_k$ iff $j, k \in \Pi_m$
- $i_j < i_k$ iff $j \in \Pi_{m_1}$ $k \in \Pi_{m_2}$ and $m_1 < m_2$.

Example $\mathbf{M}_{2/13} = x_2 x_1 x_2 + x_3 x_1 x_3 \ldots$

NC quasisymmetric Schurs

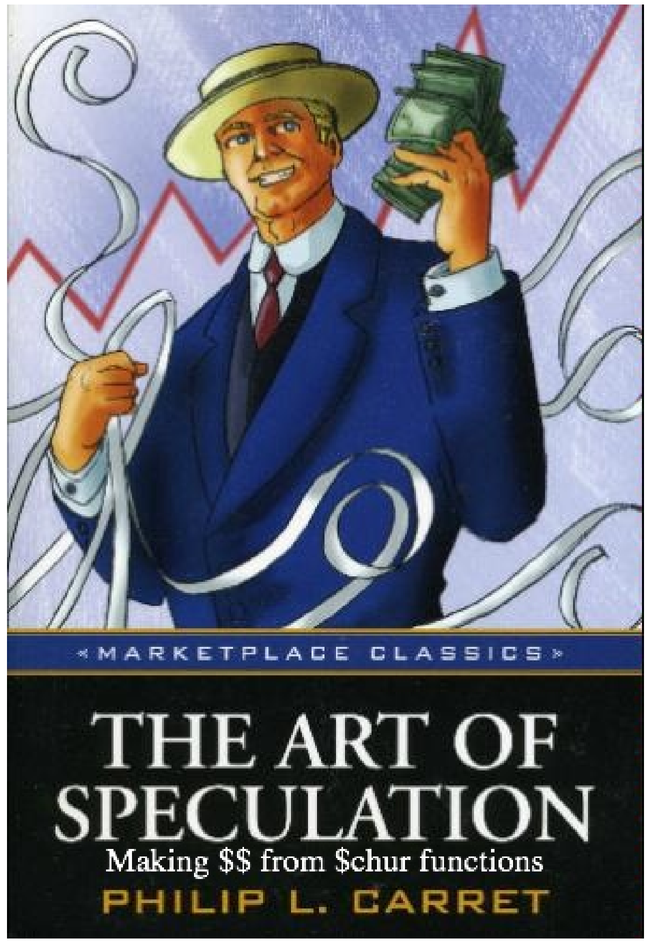
Let

$$S_\alpha^{RS} = \sum_{T \in CT(\alpha)} x^{\dot{T}} = \sum_{\beta} \beta! K_{\alpha\beta} \sum_{\alpha(\Pi)=\beta} \mathbf{M}_\Pi$$

Furthermore

$$\begin{array}{rccc} & S_\lambda^{RS} & = & \sum_{\lambda(\alpha)=\lambda} S_\alpha^{RS} \\ Rosas{-}Sagan \rightsquigarrow & \chi \big\downarrow & & \big\downarrow \chi \\ & n!\ s_\lambda & = & n!\ \sum_{\lambda(\alpha)=\lambda} \mathcal{S}_\alpha \end{array}$$

Speculation



Further avenues

Other properties:

- Jacobi-Trudi, Giambelli (quasi-) determinantal formulae?
- Representation theoretic interpretation?
- $\mathcal{L}_{\mathcal{C}}$ shellable etc?
- Normal or Kronecker (inner) product?

Other applications:

- Skew/quasisymmetric H-L, Macd. polynomials?
- Skew Demazure atoms, characters?
- Product of Schubert polynomials?
- Impact on descent algebras, different types?

Further reading

• Quasisymmetric Schur functions (with Haglund, Luoto and Mason), J. Combin. Theory Ser. A (2009).

• Refinements of the Littlewood-Richardson rule (with Haglund, Luoto and Mason), Trans. Amer. Math. Soc. (2009).

Thank you!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That's all Folks!