Euler enumeration and Balanced and Bruhat graphs.

Richard Ehrenborg, UK, IAS + Princeton
Mark Goresky, IAS
Margaret Readdy, UK, IAS + Princeton.

Euler enumeration.

with. Mark Goresky + Richard Ehrenborg.

P n-dim'l polytope

The **f-vector** $(f_0, \text{—}, f_{n-1})$

$f_i = \#$ i-dim'l faces.

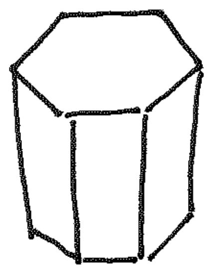
[Steinitz 1906] Characterized f-vectors of 3-dim'l polytopes

Open Q: characterize f-vectors of n-dim'l polytopes, $n \geq 4$

[Stanley 1978; Billera-Lee 1980]. Done for simplicial polytopes.

P, n-dim'l polytope

The <u>flag f-vector</u> f_S



$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_T,$$

the <u>flag h-vector</u>

S	f_S	h_S	u_S
$\emptyset$	1	1	aaa
0	12	11	baa
1	18	17	aba
2	8	7	aab
01	36	7	bba
02	36	17	bab
12	36	11	abb
012	72	1	bbb

[Stanley] $h_S = h_{\bar{S}}$

The ab-index

$$\Psi(P) = \sum_{S}{}' h_S \cdot w_S.$$

ex $\Psi($  $) = 1aaa + 11baa + 17aba + 7aab$
$+ 7bba + 17bab + 11abb + 1bbb$

$$= (a+b)^3 + 10(baa + aba + bab + abb) + 6(aba + aab + bba + bab)$$

$$= (a+b)^3 + 10(ab+ba)(a+b) + 6(a+b)(ab+ba)$$

$$= c^3 + 10dc + 6cd,$$

where $c = a+b$, $d = ab+ba$. The cd-index

Theorem: [Bayer–Klapper 1991; Stanley 1994].

P polytope then $\Psi(P) \in \mathbb{Z}\langle c, d\rangle$

P Eulerian poset then $\Psi(P) \in \mathbb{Z}\langle c, d\rangle$.

Eulerian: $\mu(x,y) = (-1)^{\rho(x,y)}$ for every interval $[x,y]$ in a graded poset P.

Equivalently, in each non-trivial interval $[x,y]$:

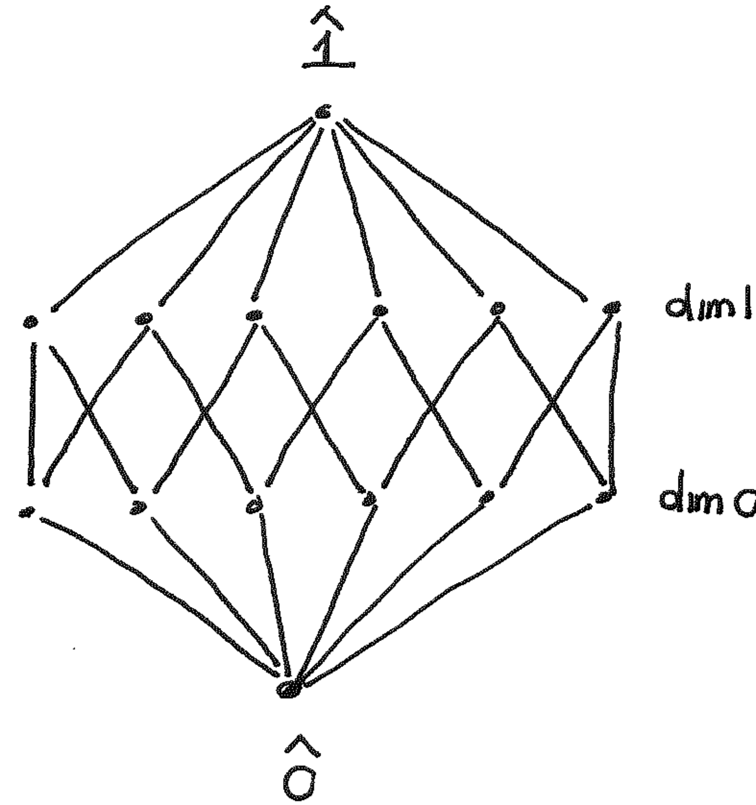
elts of even rank = # elts of odd rank.

A brief cd-history.

[Bayer - Billera 1985]	Generalized Dehn - Sommerville relations.
[Bayer - Klapper 1991]	Ψ removes all linear relations among flag vector entries.
[Stanley 1994].	$\Psi \geq 0$ for $\mathcal{L}$(polytope), more generally, S-shellable face poset of regular CW-complex.
[Purtill 1993].	n-simplex n-cube $\Longleftrightarrow$ André & signed André perms.
[Ehrenborg - R 1998]	coalgebraic techniques.

ex. The n-gon $(n \geq 2)$

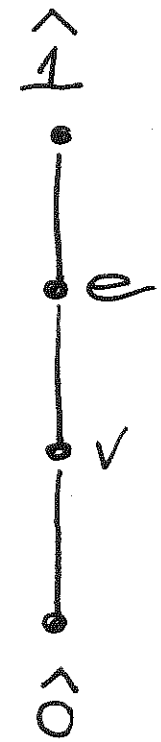
S	f_S	h_S	w_S
$\emptyset$	1	1	aa
0	n	$n-1$	ba
1	n	$n-1$	ab
01	$2n$	1	bb.



$\Psi($  $) = c^2 + (n-2)d.$

ex 1-gon

S	f_S	h_S
$\emptyset$	1	1
0	1	0
1	1	0
01	1	0



Not Eulerian.

Try again ...

$\mathrm{link}_e(v) = \bullet \ \bullet$

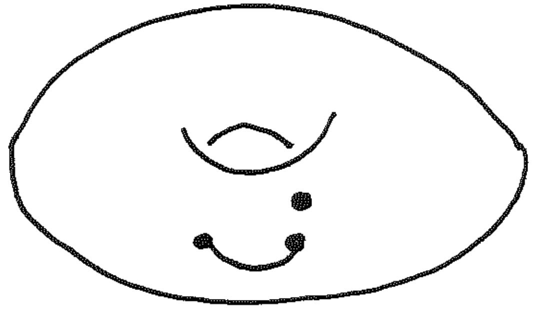
$\chi(\bullet \ \bullet) = 2$, the Euler characteristic.

S	$\overline{f}_S$	$\overline{h}_S = \sum_{T \subseteq S} (-1)^{\lvert S-T \rvert}\, \overline{f}_T$
$\emptyset$	1	1
0	1	0
1	1	0
01	2	1

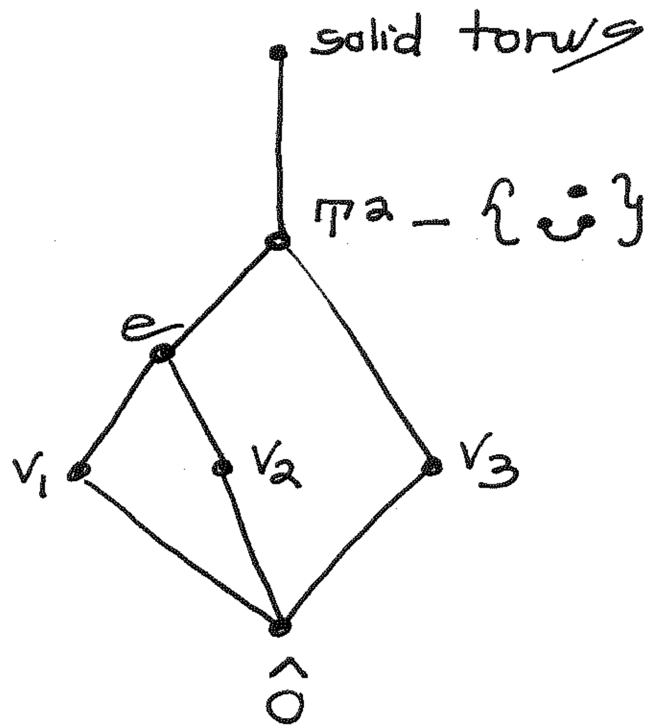
$\Phi($  $) = aa + bb$

$= c^2 - d.$

ex.



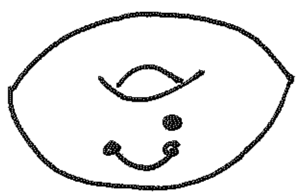
Face poset



Chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$
in the face poset weighted by.

$$\bar{\eta}(c) = \chi(x_1) \cdot \chi(\mathrm{link}_{x_2}(x_1)) \cdots \chi(\mathrm{link}_{x_k}(x_{k-1}))$$

ex. (cont'd).



S	$\bar{f}_S$	$\bar{h}_S$	$3dc$	$-2cd$
$\emptyset$	0	0	0	0
0	3	3	3	0
1	1	1	3	-2
2	-2	-2	0	-2
01	2	-2	0	-2
02	2	1	3	-2
12	2	3	3	0
012	4	0	0	0

$\Phi($  $) = 3dc - 2cd.$

These are examples of
<u>Whitney stratifications</u>

Subdivide space into strata:

$$W = \dot{\bigcup_{X \in P}} X$$

<u>Condition of the frontier</u>:

$X \cap \overline{Y} \neq \emptyset \iff X \subseteq \overline{Y} \iff X \leq_P Y$ in face poset P.

Whitney conditions A + B:
No fractal behavior
No infinite wiggling ex. $\sqrt{x} \cdot \sin\left(\frac{1}{\sqrt{x}}\right)$

$\Rightarrow$ The links are well-defined.

The Fine Print

Definition Let W be a closed subset of a smooth manifold M, and suppose W can be written as a locally finite disjoint union

$$W = \bigcup_{X \in \mathcal{P}} X$$

where $\mathcal{P}$ is a poset. Furthermore, suppose each $X \in \mathcal{P}$ is a locally closed subset of W satisfying the *condition of the frontier*:

$$X \cap \overline{Y} \neq \emptyset \Longleftrightarrow X \subseteq \overline{Y} \Longleftrightarrow X \leq_{\mathcal{P}} Y.$$

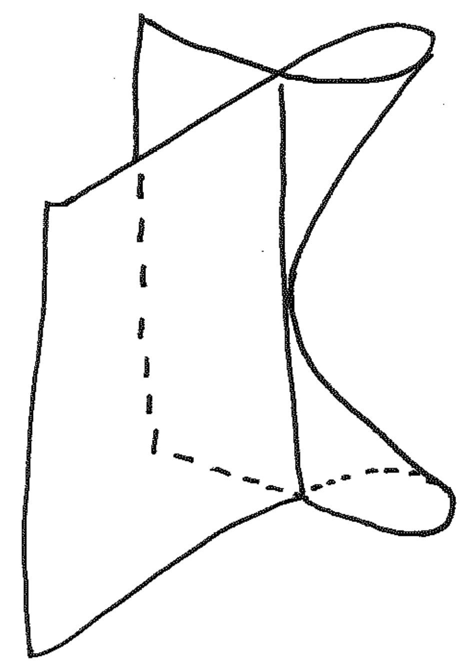
This implies the closure of each stratum is a union of strata. We say W is a *Whitney stratification* if

1. Each $X \in \mathcal{P}$ is a locally closed smooth submanifold of M (not necessarily connected).
2. If $X <_{\mathcal{P}} Y$ then Whitney's conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to x. Also assume that (with respect to some local coordinate system on the manifold M) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line ℓ and the tangent planes $T_{y_i}Y$ converge to some limiting plane τ. Then the inclusions

   $$\text{(A) } T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B) } \ell \subseteq \tau$$

   hold.

ex. The Whitney cusp.



Whitney stratifications (their face posets) are examples of ...

A <u>quasi-graded poset</u> $(P, \rho, \bar{\zeta})$ consists of

i. P finite poset with $\hat{0}$ & $\hat{1}$ (not necessarily graded)

ii. $\rho : P \to \mathbb{N}$ order-preserving ($x < y \Rightarrow \rho(x) < \rho(y)$)

iii. $\bar{\zeta} \in I(P)$, the weighted zeta function satisfying $\bar{\zeta}(x,x) = 1 \quad \forall x \in P$.

def. $(P, \rho, \bar{\zeta})$ <u>Eulerian</u> if

$$\sum_{x \le y \le z} (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x,y) \cdot \bar{\zeta}(y,z) = \delta_{x,z}.$$

Remark: $\bar{\zeta} = \zeta$ gives the <u>classical Eulerian condition</u>

$$\sum_{x \le y \le z} (-1)^{\rho(x,y)} = \delta_{x,z}.$$

Define

$$\Phi(P, \rho, \bar{\xi}) = \sum_{S} \bar{h}_S \cdot w_S$$

with.

$$\bar{\xi}(c) = \bar{\xi}(x_0, x_1)\, \bar{\xi}(x_1, x_2) \cdots \bar{\xi}(x_{k-1}, x_k).$$

for a chain c: $\underset{\hat{0}.}{\overset{\|}{x_0}} < x_1 < \cdots < x_k = \hat{1}$

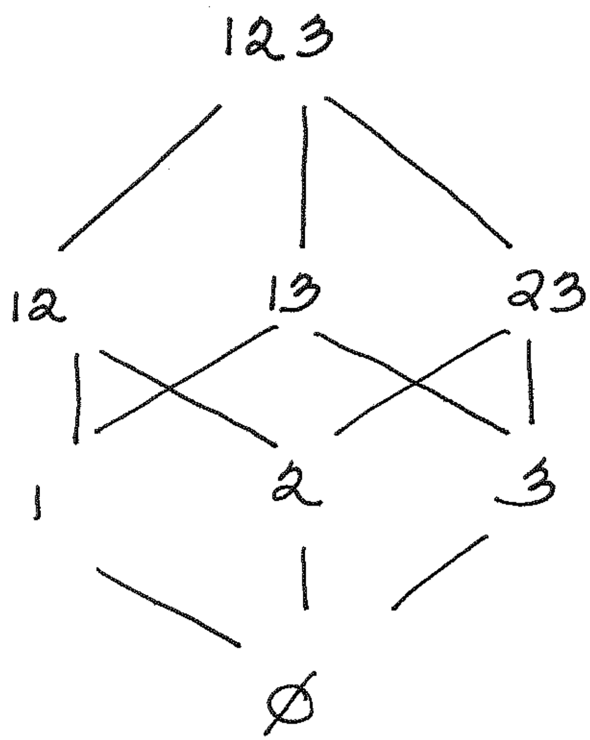
Theorem: $(P, \rho, \bar{\xi})$ an Eulerian quasi-graded poset. Then
$$\Psi(P, \rho, \bar{\xi}) \in \mathbb{Z}\langle c, d\rangle.$$

Theorem: M manifold with a Whitney stratified boundary, Then the face poset is quasi-graded + Eulerian, where
$$\rho(x) = \dim(x) + 1.$$
$$\bar{\xi}(x, y) = \chi(\operatorname{link}_y(x)).$$

Balanced and Bruhat graphs.

with Richard Ehrenborg

P graded poset.



$= \mathcal{L}($  $)$

$$\sum_{T \subseteq S} (-1)^{|S-T|} f_T$$

$$\|$$

S	f_S	h_S	u_S
$\emptyset$	1	1	aa
1	3	2	ba
2	3	2	ab
12	6	1	bb.

$$\Psi = aa + 2ba + 2ab + bb$$

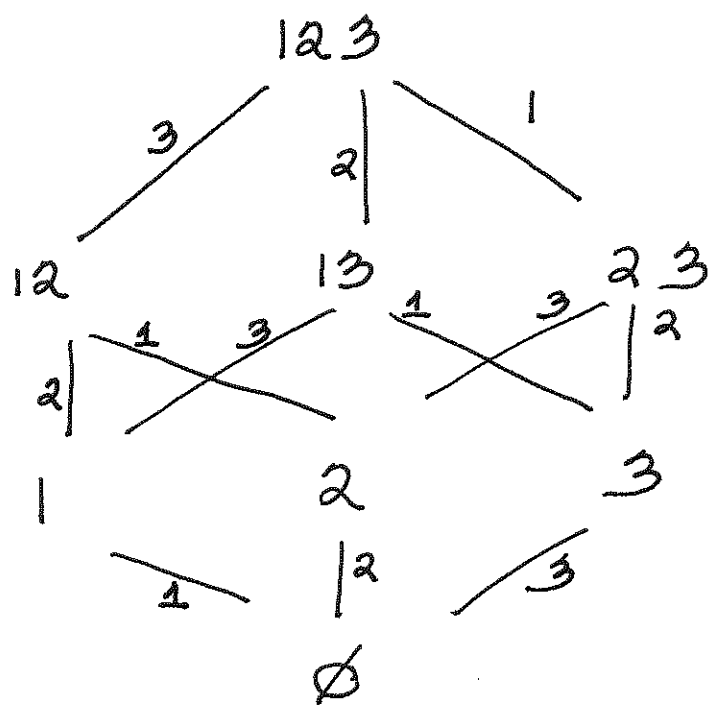
$$= (a+b)^2 + (ab+ba)$$

$$= c^2 + d.$$

Poset labeling

def. We say $\lambda : \mathcal{E}(P) \to \Lambda$ is an R-labeling if in each interval there exists a unique rising chain

ex. (classical).
For Boolean algebra B_n, label edge $A \subsetneq B$, $|A| = |B| - 1$ (i.e., $A \lessdot B$) by $B - A$:

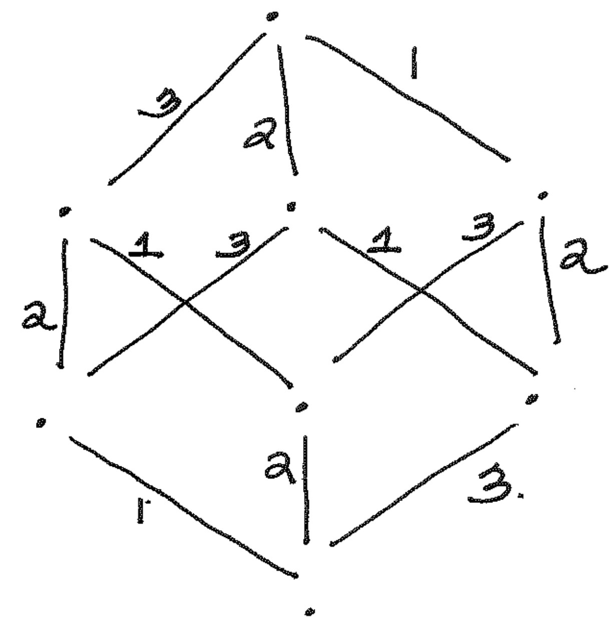


Theorem: [Björner - Stanley].
P graded poset with R-labeling λ.
Then

$$h_S = \text{\# maximal chains in } P \text{ having descent set } S \text{ (wrt R-labeling } \lambda).$$

ex.

c	$w(c)$
123	aa
132	ab
213	ba
231	ab
312	ba
321	bb.



$\Psi = aa + 2ab + 2ba + bb.$

Corollary: P graded poset with R-labeling. Then the ab-index is

$$\Psi(P) = \sum_{\substack{c \text{ max'l} \\ \text{chain} \\ \text{in } P.}} w(c)$$

Digraphs.

G acyclic digraph

Allow multiple edges.

Unique source + sink.

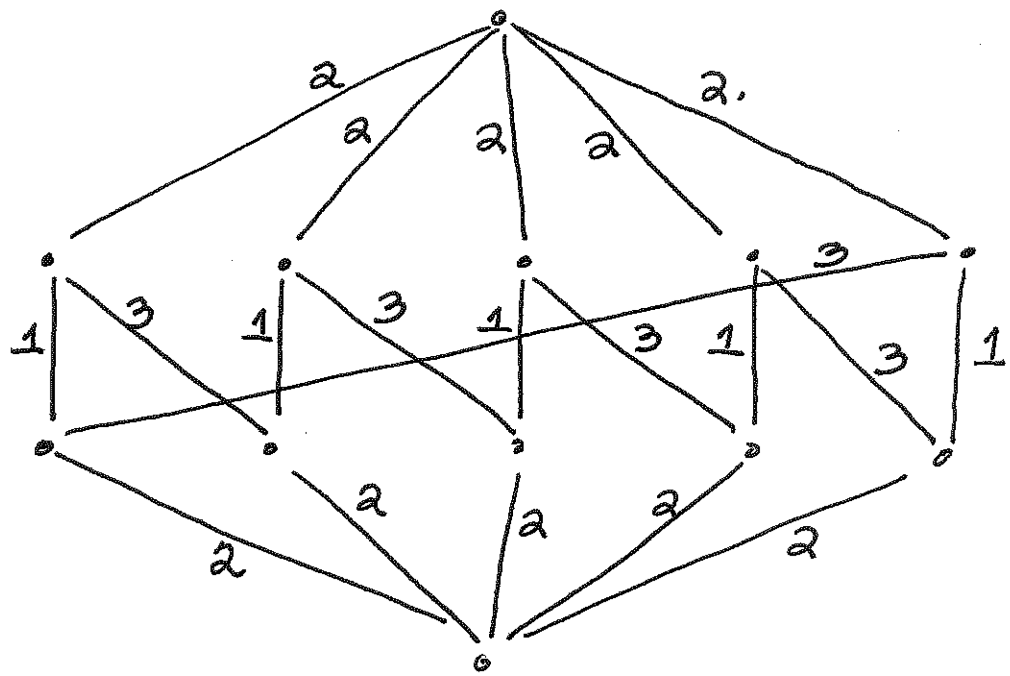
View G as a poset!

$$x \leq y. \iff \exists \text{ path } x \to \cdots \to y.$$

$$[x,y] \iff \{v \in V(G) ;\ x \to \cdots \to v \to \cdots \to y\}.$$

Examples.

ex. $\mathcal{L}$(n-gon)



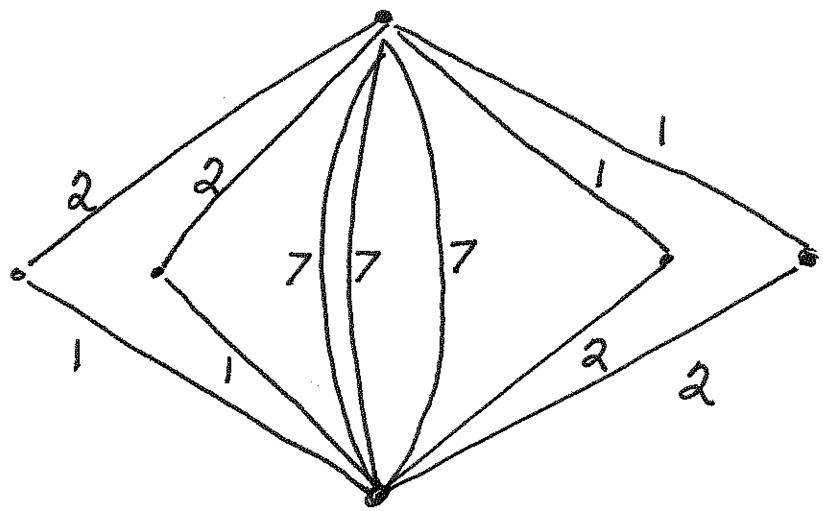
(direct all edges upwards).

5 max'l chains labeled 212

5 ——————— 232.

$$\sum_c{}' \, w(c) = 5ba + 5ab = 5d.$$

ex.



$$\Phi(G) = 2a + 2b + 3$$

$$= 2c + 3$$

Q: When can we write $\Psi([x,y])$ as a cd-index ?

Define

$$\tilde{r}_{[x,y]} = \sum_{\substack{\text{all max'l} \\ \text{rising paths/chains} \\ \text{from } x \text{ to } y}} q^{\ell(c)-1}$$

$$\tilde{f}_{[x,y]} = \sum_{\substack{\text{all max'l} \\ \text{falling ——} \\ \text{————}}} q^{\ell(c)-1}.$$

Here $\ell(c)$ = length of the chain.

Theorem: If for all intervals $[x,y]$ in G we have

$$\tilde{f}_{[x,y]}(G) = \tilde{r}_{[x,y]}(G)$$

then the ab-index of G can be written as a polynomial in c's and d's with $\mathbb{Z}$ coefficients

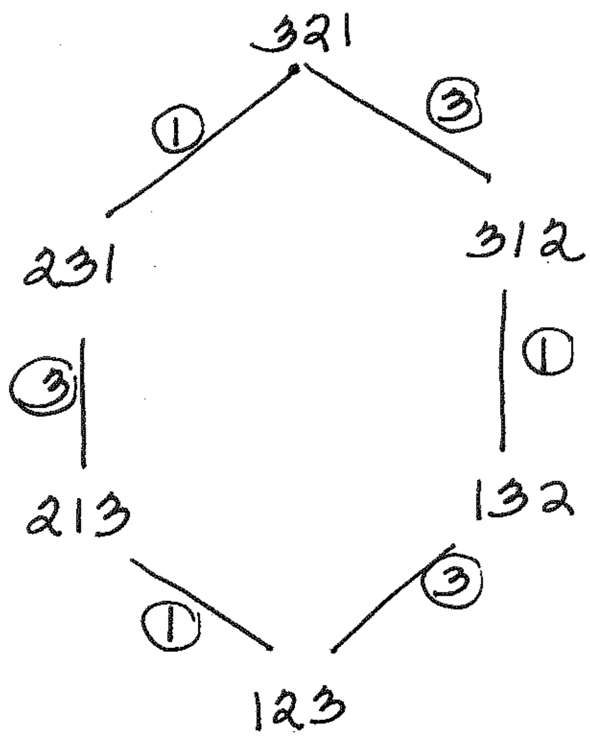
Call such labeled digraphs <u>balanced</u>.

Special case: Bruhat graphs

ex. $\mathfrak{S}_n$ is generated by

$$\{(1,2),\ (2,3),\ \ldots,\ (n-1,n)\}.$$

The weak Bruhat order (only adjacent transpositions).

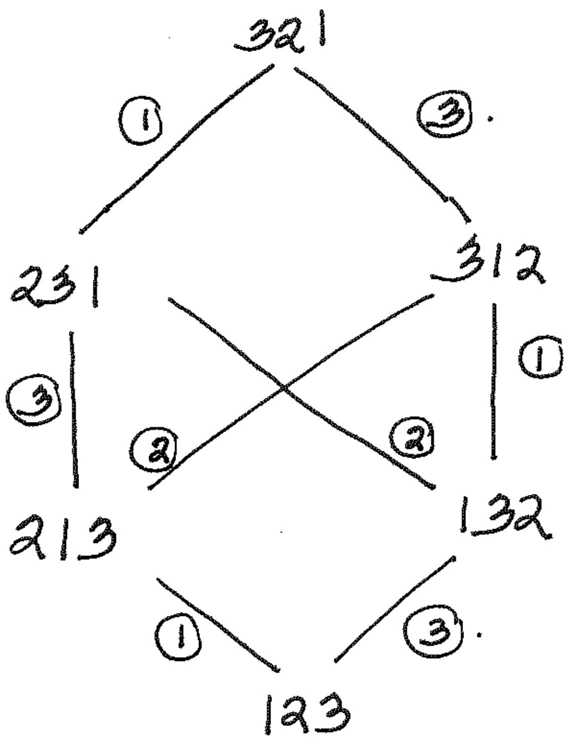


Use reflection ordering

$(1,2) < (1,3) < (2,3).$

1 2 3

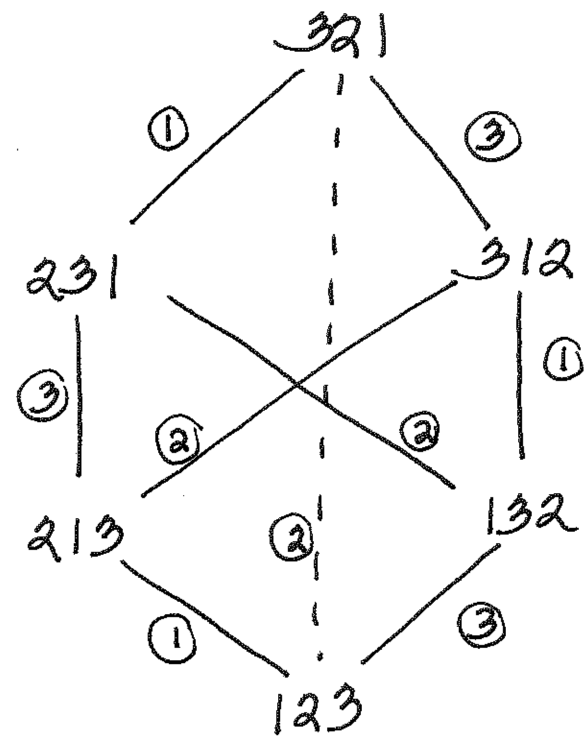
The strong Bruhat order



$(1,2) < (1,3) < (2,3)$

1 2 3.

The Bruhat graph



(allow ~~shortcuts~~).

c	u(c)
131	ab
123	aa
321	bb
313	ba
2	1.

$$\Phi = (a+b)^2 + 1$$

$$= c^2 + 1.$$

Theorem: [Billera - Brenti].

i. The "complete" cd-index exists for Coxeter groups.

<u>Proof</u>.

Hard.

Uses quasisymmetric functions & peak algebra $\boxed{}$.

Now is an easy.

Corollary: [E-R].

<u>Proof</u>.

The reflection ordering is reversible.

$\tilde{r} = \tilde{f} \implies$ cd-index exists $\boxed{}$.

Theorem: (cont'd)

ii. The top degree gives classical cd-index

Proof [E-R]

Restrict Bruhat graph to Bruhat poset. (no shortcuts)

Reflection ordering is an R-labeling ▨.

iii. Degrees of cd-terms have same parity.

Proof [E-R]

Digraph is bipartite ▨

Current Work.

①. Nonnegativity of Φ coeffs for

[Stanley]: polytopes &
S-shellable posets.

[Karu]: Gorenstein* lattices.

Is there a stratified explanation?

②. Inequalities:

[Kalai] Kalai convolution still works.

What about Ehrenborg's lifting technique?

③. $(P, \rho, \bar{\xi}) \Rightarrow$ Find W Whitney stratified space.

④. Combinatorial interpretation for cd-index coeffs.

[Purtill] n-simplex & n-cube

[Karu] operators on sheaves of v.s.

5. Develop Kazhdan–Lusztig polynomials for balanced graphs.

6. $\Psi \geq 0$ for balanced graphs.

7. When does one have an R-labeling?
(Ehrenborg–R: Not all Eulerian posets have R-labelings).

8. [Ehrenborg–Hetyei–R].
Level Eulerian posets.

Thank you!