Coalgebraic Combinatorics

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V vector space over a field $\Bbbk$.

Algebra: Product

$$\mu : V \otimes V \longrightarrow V \qquad \text{(linear map).}$$

Is associative

$$\mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu).$$

$$\begin{tikzcd}
(V \otimes V) \otimes V \arrow[r, phantom, "\cong"] \arrow[d, "\mu \otimes 1"'] & V \otimes V \otimes V \arrow[r, phantom, "\cong"] & V \otimes (V \otimes V) \arrow[d, "1 \otimes \mu"] \\
V \otimes V \arrow[dr, "\mu"'] & & V \otimes V \arrow[dl, "\mu"] \\
& V &
\end{tikzcd}$$

Coalgebra: Coproduct

$$\Delta : \quad V \longrightarrow V \otimes V \qquad \text{(linear map).}$$

Coassociativity.

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$$

$$\begin{tikzcd}
& V \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
V \otimes V \arrow[d, "\Delta \otimes 1"'] & & V \otimes V \arrow[d, "1 \otimes \Delta"] \\
(V \otimes V) \otimes V \arrow[rr, phantom, "\cong"] & & V \otimes (V \otimes V)
\end{tikzcd}$$

Sweedler notation

$$\Delta(x) = \sum_{x} x_{(1)} \otimes x_{(2)}.$$

def. [Joni – Rota]
(V, μ, Δ) is a <u>Newtonian coalgebra</u>
if it satisfies the Newtonian condition

$$\Delta \circ \mu = (1 \otimes \mu) \circ (\Delta \otimes 1) + (\mu \otimes 1) \circ (1 \otimes \Delta)$$

Sweedler:

$$\Delta(x \cdot y) = \sum_{x} x_{(1)} \otimes (x_{(2)} \cdot y) + \sum_{y} (x \cdot y_{(1)}) \otimes y_{(2)}.$$

(Gen'n of product rule for derivative)

Define linear map

$$D_v : x \longmapsto D_v(x) = \sum_{x} x_{(1)} \vee x_{(2)}$$

is a derivation on the algebra (V, μ).

$$D_v(x \cdot y) = D_v(x) \cdot y + x \cdot D_v(y).$$

Example 1: Coalgebra on $k[x]$
[Hirschorn – Raphael].

Δ linear map.

$$\Delta(x^n) = \sum_{i+j=n-1} x^i \otimes x^j$$

Extend to $k[x]$ by linearity.

Lemma: Δ is coassociative.

Proof

$$(\Delta \otimes 1) \circ \Delta(x^n) = (\Delta \otimes 1) \sum_{i+j=n-1} x^i \otimes x^j$$

$$= \sum_{i'+i''+j=n-2} x^{i'} \otimes x^{i''} \otimes x^j$$

$$= \sum_{i+j+k=n-2} x^i \otimes x^j \otimes x^k$$

$$= \dots = (1 \otimes \Delta)\,\Delta(x^n) \quad \boxtimes$$

~~Ig~~ Newtonian.

$$\Delta(x^{i}\cdot y^{j}) = \sum_{j_1+j_2=j-1} x^{i}\cdot y^{j_1}\otimes y^{j_2}$$

$$+ \sum_{i_1+i_2=i-1} x^{i_1}\otimes x^{i_2}\cdot y^{j}.$$

Example 2: $\mathcal{A} = k\langle a,b\rangle$ = polynomial algebra in noncomm. vars $a+b$

[Ehrenborg - R].

Product = usual.

Coproduct:

$$\Delta(v_1 \cdots v_n) = \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n .$$

Extend to all of $\mathcal{A}$ by linearity.

ex. $\Delta(abba) = 1 \otimes bba + a \otimes ba + ab \otimes a + abb \otimes 1.$

Lemma: ① $(\mathcal{A}, \cdot, \Delta)$ is a Newtonian coalgebra

② $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ is graded.

($\mathcal{A}_n$ is spanned by degree n monomials)

$\dim(\mathcal{A}_n) = 2^n$

③ $\mathcal{A}_i \cdot \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$.

$$\underbrace{\Delta(\mathcal{A}_n)}_{\dim = 2^n - 1} \subseteq \underbrace{\bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j}_{\dim = n \cdot 2^{n-1}}$$

④ The kernel of Δ is 1-dim'l & spanned by $(a-b)^n$.

Example 3: $\mathcal{P}$ = vector space of all types of graded posets w/ $\hat{0} \neq \hat{1}$ over field k

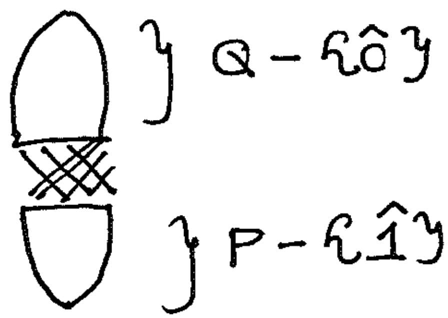
[Ehrenborg–Hetyei].

Define coproduct

$$\Delta(\overline{P}) = \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} [\hat{0}, x] \otimes [x, 1].$$

Define star product

$P * Q =$

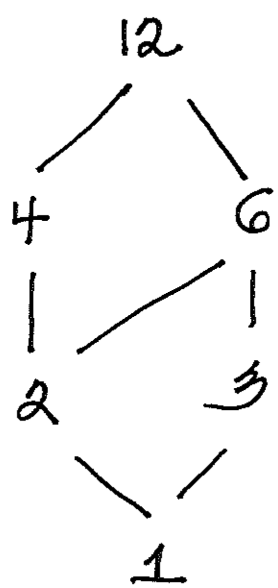


Theorem: [E–H]

$(\mathcal{P}, *, \Delta)$ is a Newtonian coalgebra.

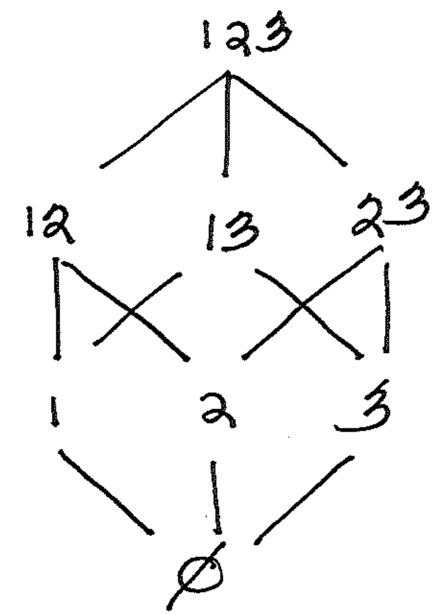
ex posets.

Divisor lattice.



graded by prime decomp.

B_n, Boolean algebra



graded by $|S|$.

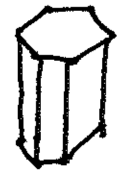
Application: Polytopes

A polytope is the convex hull of a finite # of vertices in $\mathbb{R}^n$.

or

... is the bounded intersection of a finite # of closed half-spaces in $\mathbb{R}^n$.

Flag vectors.

ex. Prism (⬡) = 

S	f_S	$h_S = \sum_{T \subseteq S} (-1)^{\lvert S-T \rvert} f_T$	
$\emptyset$	1	1	aaa
0	12	11	baa
1	18	17	aba
2	8	7	aab
01	36	7	bba
02	36	17	bab
12	36	11	abb
012	72	1	bbb.

[Stanley] $h_S = h_{\bar{S}}$

The ab-index

$$\Psi(P) = \sum_{S} h_S \cdot w_S$$

$\Psi($  $)$ $= 1\,aaa + 11\,baa + 17\,aba + 7\,aab$

$+ 7\,bba + 17\,bab + 11\,abb + 1\,bbb$

$= (a+b)^3 + 10\,baa + 16\,aba + 6\,aab$

$+ 6\,bba + 16\,bab + 10\,abb$

$= (a+b)^3 + 6(a+b)(ab+ba) + 10\,(ab+ba)(a+b).$

Let

$c = a+b$

$d = ab+ba$

Then

$\Psi($  $)$ $= c^3 + 6cd + 10dc.$

The cd-index.

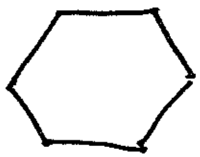
[Bayer-Klapper] The ab-index of a polytope, more generally, a graded Eulerian poset, can be written as a cd-index: $\Psi \in \mathbb{Z}\langle c,d\rangle$.

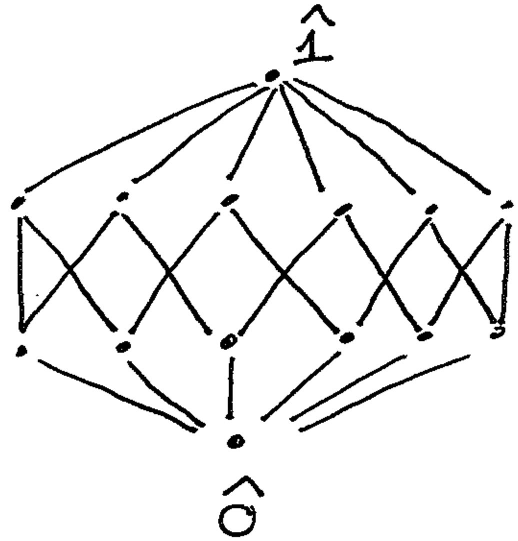
[Bayer-Billera] The cd-index Ψ removes all of the linear redundancies among the flag vector entries.

[Stanley]. $\Psi \geq 0$ for $\mathcal{L}$(polytopes), more generally, spherically-shellable posets.

Open: Find combinatorial interpretation of the coeffs of the cd-index.

ex. 6-gon. $\mathcal{L}$(6-gon)





edges (dim 1).

vertices (dim 0)

S	f_S	h_S	w_S
$\emptyset$	1	1	aa
0	6	5	ba
1	6	5	ab
01	12	1	bb

Ψ(⬡) $= c^2 + 4d.$

$\Phi(⬡) = c^2 + 4d$

$\Phi(\text{Prism}(⬡)) = c^3 + 6cd + 10dc.$

Want: $= (c^2+4d)\cdot c + 6(cd+dc).$

Let $D(c) = 2d$

$D(d) = cd + dc.$

Then $D(c \cdot c) = c \cdot D(c) + D(c) \cdot c$

$= c \cdot 2d + 2d \cdot c.$

$= 2cd + 2dc.$

Theorem: [Ehrenborg - R]

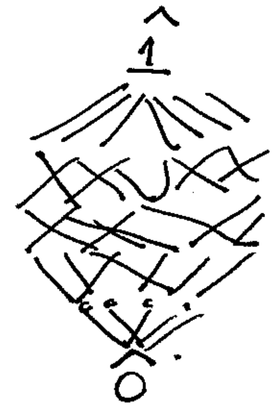
$$\Psi(\mathrm{Pyr}(P)) = \Psi(P)\cdot c + D(\Psi(P))$$

where D is the derivation

$$D(c) = 2d$$
$$D(d) = cd + dc.$$

Proof Ingredients

① . P polytope of dim . $\longleftrightarrow$ $\mathcal{L}(P)$



$\mathcal{L}(P)$ graded of rank $+1$.

$f_S \longleftrightarrow$ # chains $\hat{0} < x_1 < \dots < x_k < \hat{1}$
with $S = \{\rho(x_1), \dots, \rho(x_k)\}$.

$h_S \longleftrightarrow$ # chains above with weight
$wt(\hat{0} < x_1 < \dots < x_k < \hat{1}) = z_1 \cdots z_n$

$$z_i = \begin{cases} b & \text{if } i \in \{\rho(x_1), \dots, \rho(x_k)\} \\ a-b & \text{otherwise.} \end{cases}$$

Then

$$\underline{\Psi}(\mathcal{L}(P)) = \sum_{\substack{\text{chains} \\ c}} wt(c).$$

ex. hexagon.

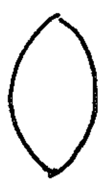
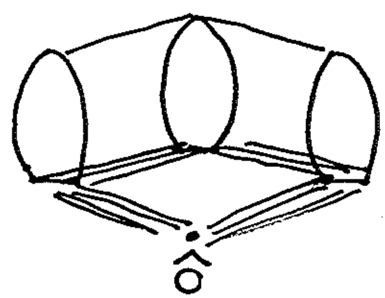
$\underline{c}$	$\underline{wt(c)}$	
$\hat{0} < \hat{1}$	$(a-b)^2$	$a^2 - ab - ba + b^2$
$\hat{0} < v < \hat{1}$	$6b \cdot (a-b)$	$6ba - 6bb$
$\hat{0} < e < \hat{1}$	$6\ (a-b) \cdot b$	$6ab - 6bb$
$\hat{0} < v < e < \hat{1}$	$12\ bb.$	$12bb$

$$a^2 + 5ab + 5ba + b^2.$$

graded poset

②. $$\Phi(\mathrm{Prism}(P)) = \Phi(P)\cdot(a+b) + \sum_{\substack{x\in P \\ \hat{0}<x<}} \Phi([\hat{0},x])\cdot(ab+ba)\cdot\Phi([x,\hat{1}]).$$

(via a careful chain argument).

Prism () = 

③ Theorem: [Ehrenborg – R]

The ab-index is a Newtonian coalgebra homomorphism from $\mathcal{P}$ to $\mathcal{A} = k\langle a, b\rangle$ with

$$\Psi(\mathbf{1}) = 1$$

$$\Psi(P * Q) = \Psi(P) \cdot \Psi(Q)$$

$$\Delta(\Psi(P)) = \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \otimes \Psi([x, \hat{1}])$$

③′ Corollary: [Ehrenborg – R].

The cd-index is a Newtonian coalgebra homomorphism from

$\mathcal{E}$ = linear space of graded Eulerian posets

to $k\langle c, d\rangle$.

④. The (miracle) coproduct on $\mathfrak{A}$.

$$\left.\begin{aligned} \Delta(a) &= 1 \otimes 1 \\ \Delta(b) &= 1 \otimes 1 \end{aligned}\right\} \quad \underset{\Delta(c)}{\overset{\Delta(a+b)}{\shortparallel}} = 2 \cdot 1 \otimes 1$$

$$\begin{aligned} \Delta(ab+ba) &= a\,\Delta(b) + \Delta(a)\cdot b \\ &\quad + b\,\Delta(a) + \Delta(b)\cdot a \end{aligned}$$

$\shortparallel$

$\Delta(d)$

$$\begin{aligned} &= a\cdot 1\otimes 1 + 1\otimes b \\ &\quad + b\otimes 1 + 1\otimes a \\ &= c\otimes 1 + 1\otimes c \end{aligned}$$

(3). The derivation D on $\mathcal{A}$. ($+ \varepsilon$).

$$\Delta(c) = 2 \cdot 1 \otimes 1$$

$$\Delta(d) = c \otimes 1 + 1 \otimes c.$$

$$\Rightarrow D(c) = 2d$$

$$D(d) = cd + dc.$$

$\blacksquare$

Thank you!